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TOPICAL REVIEW

Bethe ansatz solvable multi-chain quantum systems

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Abstract

In this article we review recent developments in the one-dimensional Bethe ansatz solvable multi-chain quantum models. The algebraic version of the Bethe ansatz (the quantum inverse scattering method) permits us to construct new families of integrable Hamiltonians using simple generalizations of the well known constructions of the single-chain model. First we consider the easiest example ('basic' model) of this class of models: the antiferromagnetic two-chain spin-$\frac{1}{2}$ model with the nearest-neighbour and next-nearest-neighbour spin-frustrating interactions (zigzag chain). We show how the algebra of the quantum inverse scattering method works for this model, and what are the important features of the Hamiltonian (which reveal the topological properties of two dimensions together with the one-dimensional properties). We consider the solution of the Bethe ansatz for the ground state (in particular, commensurate–incommensurate quantum phase transitions present due to competing spin-frustrating interactions are discussed) and construct the thermal Bethe ansatz (in the form of the 'quantum transfer matrix') for this model. Then possible generalizations of the basic model are considered: an inclusion of a magnetic anisotropy, higher-spin representations (including the important case of a quantum ferrimagnet), the multi-chain case, internal degrees of freedom of particles at each site, etc. We observe the similarities and differences between this class of models and related exactly solvable models: other groups of multi-chain lattice models, quantum field theory models and magnetic impurity (Kondo-like) models. Finally, the behaviour of non-integrable (less constrained) multi-chain quantum models is discussed.

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1. Introduction

Approximate methods are commonly used in theoretical physics of many-body systems. Probably the best known approximate methods are: perturbation theories, mean-field-like theories, variational approaches and critical point approximations in their various realizations. However, exactly solvable models take their own special place among other models of theoretical physics. There are several situations in which studies of such models are very important and sometimes necessary. For example, one needs to find an exact solution if a theory has no *small parameter*, or if there is no *order parameter* (i.e. one cannot use a mean-field-like approach), or if one needs to find some characteristics in a non-perturbative way (e.g. such phenomena as a renormalization, asymptotic freedom, onset of a mass via the regularization parameters in the field theory, etc). On the other hand, exactly solvable (integrable) models are important in their own right—the exact knowledge of some characteristics of one system yields a possibility of using other powerful approximate methods to study more realistic physical models. Note that in recent years there have appeared many real systems whose behaviour can be described by exactly solvable models. The great majority of exactly solvable models of many-body physics can be divided into two large classes: two-dimensional classical statistical models and one-dimensional quantum models. Examples of the first class are the well known Ising model (with a great number of its generalizations), ice-like models and vertex models [1]. The seminal model for the second class of exactly solvable models is the spin-$\frac{1}{2}$ Heisenberg chain with nearest-neighbour (NN) antiferromagnetic (AF) interactions. The method of solving the Schrödinger equation for this model and related class of models was pioneered by Bethe [2] and is now known as the *Bethe ansatz*. It was recognized later (first of all due to very important works of Yang and Baxter) that the Bethe ansatz can be applied to models in which the scattering between (quasi-) particles is only *elastic* (and, obviously, non-dispersive). Mathematically the condition of elasticity was formulated in the form of the well known *Yang–Baxter equations* (YBEs). Suppose one has several (interacting) particles. Then the YBE for their *two-particle scattering matrices* can be written as [1]

$$\hat{S}_{12}(u_1, u_2)\hat{S}_{13}(u_1, u_3)\hat{S}_{23}(u_2, u_3) = \hat{S}_{23}(u_2, u_3)\hat{S}_{13}(u_1, u_3)\hat{S}_{12}(u_1, u_2) \qquad (1)$$

where indices ($j = 1, 2, 3$) denote particles with quantum numbers (called *rapidities*) u_j (they are related to momenta of particles). The YBEs mean that each scattering process produces only a phase shift (and no reflection) and the order of scatterings is not essential. Rapidities satisfy some set of transcendental equations (known as Bethe ansatz equations, BAE) [2]. Eigenfunctions and eigenvalues of Schrödinger equations are determined (usually in a unique way) by that set of rapidities [2]. Later it was recognized that for some exactly solvable models of the first class (two-dimensional statistical models) their *statistical weights* also satisfy YBE [1]. This knowledge permitted the Leningrad (St. Petersburg) group headed by Faddeev to create the algebraic version of the Bethe ansatz called the quantum inverse scattering method (QISM) [3, 4]. This method is based on the mapping of a one-dimensional quantum problem onto the associated two-dimensional statistical problem (as well as being closely related to the classical inverse scattering method [3, 4]).

In the following we shall review a recently introduced class of exactly solvable models (which is the obvious generalization of the known Bethe ansatz solvable models). We point out that our review cannot be complete (i.e. we only refer to those papers which we consider relevant for the topic of the article—we would like to apologize for not citing papers of other authors probably related to the topic). The structure of this review article is as follows. In the remainder of the introduction we briefly recall the main steps of the algebraic Bethe ansatz. In section 2 we consider the 'basic' model of our review: the integrable model with several

different couplings between particles—the two-chain quantum spin model. The ground-state properties, low-lying excitations, conformal behaviour of the correlation functions and the thermodynamics are discussed. In section 3 we consider possible generalizations of the basic model. Section 4 is devoted to the review of other classes of related exactly solvable models. We discuss the special case of non-Hermitian Hamiltonians, which also belongs to the considered class of integrable models, in section 5. In section 6 we show how the knowledge of exact characteristics of this class of models can be used for the study of more realistic, less constrained (non-integrable) multi-chain quantum models. Concluding remarks are given in section 7.

Let us recall how the QISM scheme works for the simplest example of the Heisenberg spin-$\frac{1}{2}$ chain (HC, sometimes called the XXX chain) of length L with the Hamiltonian $\mathcal{H}_{\rm HC} = J\sum_j \boldsymbol{S}_j\boldsymbol{S}_{j+1}$. One can start with an *R-matrix*, which depends on the *spectral parameter* u. For the HC the R-matrix has the form

$$R(u) = [\mathrm{i} - a(u)]\hat{I} + a(u)\hat{P} \tag{2}$$

where $a(u) = u/(u+\mathrm{i})$, $\hat{I}^{\gamma,\nu}_{\alpha,\beta} = \delta_{\alpha,\beta}\delta_{\gamma,\nu}$ is the identity operator and $\hat{P}^{\gamma,\nu}_{\alpha,\beta} = \delta_{\alpha,\nu}\delta_{\gamma,\beta}$ is the permutation operator. The R-matrices satisfy the YBE

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{3}$$

where the lower indices denote the Hilbert spaces in which the R matrix acts. We may consider spaces 1 and 2 as 'quantum spaces' of particles of the system and 3 as the 'auxiliary space' of some additional particle. The Lax L-operators on site l (whose quantum space corresponds to the Hilbert space of the lth site of the HC), $L(u) = \hat{P}R(u)$, also satisfy the YBE. These L-operators are direct products of the two-particle scattering matrix acting on the lth site of the HC and the auxiliary space with unit operators for the Hilbert spaces of the other sites. Let us define the *monodromy operator* as the ordered product of L-operators

$$T_L(u) = L_L(u)L_{L-1}(u)\cdots L_1(u). \tag{4}$$

Monodromy shows the result of scatterings of an auxiliary particle off all L particles in the system. By construction the monodromy matrices satisfy the YBE with the R-matrix. The trace of the monodromy matrix $\tau(u) = \mathrm{Tr}[T_L(u)]$ (taken over the auxiliary space) is the *transfer matrix* of the associated two-dimensional statistical vertex problem. As a consequence of YBE for monodromies, transfer matrices with different spectral parameters mutually commute. This means that the transfer matrix can serve as a generating functional of the (infinite) number of conserved quantities, like a Hamiltonian, an operator of the total momentum, etc. For example, the Hamiltonian of the HC with NN couplings can be constructed by taking the first derivative of the logarithm of the transfer matrix at zero spectral parameter

$$\mathcal{H}_{\rm HC} = \frac{\mathrm{i}}{2}J\left.\frac{\partial \ln\tau(u)}{\partial u}\right|_{u=0} = \frac{J}{2}\sum_{l=1}^{L}P_{l,l+1} - \frac{LJ}{4} \tag{5}$$

where $P_{l,l+1} = 2\boldsymbol{S}_l\boldsymbol{S}_{l+1} + \frac{1}{2}$ is the permutation operator for sites l and $l+1$ and J is the exchange constant. In principle, it is possible to construct an infinite number of conservation laws (integrals of motion, which commute with the transfer matrix and mutually); this means the exact integrability of the problem.

Now, the goal of the QISM is to find eigenfunctions and eigenvalues of the transfer matrix. Then the eigenvalues for the conservation laws will be calculated from eigenvalues of the transfer matrix. We choose the vacuum state $|0\rangle_k$ (in the Hilbert quantum space of the kth site) as the state with spin up. (Note that for systems without a possibility of introducing a vacuum state one has to use, e.g., the *functional* Bethe ansatz; see, e.g., [1].) One can represent the monodromy matrix $T_L(u)$ as the 2×2 operator-valued matrix $\hat{A}_{ij}$ with the trace

$\tau(u) = \hat{A}_{11} + \hat{A}_{22}$. The action of the monodromy matrix on the vacuum state $|0\rangle = \otimes_{k=1}^{L}|0\rangle_k$ has a 'triangular' form (with $\hat{A}_{21}|0\rangle = 0$). Another non-diagonal element can serve as a 'creation operator' with respect to the vacuum state; i.e., one can use the following form of eigenstates: $|\{\lambda\}_{\alpha=1}^{M}\rangle = \prod_{\alpha=1}^{M} B(\lambda_\alpha)|0\rangle$, where $B = \hat{A}_{12}$ (M is the number of down spins). The commutation relations between the elements of the monodromy matrix follow from the YBE. The action of the transfer matrix on such a state produces the state multiplied by the eigenvalue (as a function of the spectral parameter u and rapidities λ_α) and some other 'unwanted' terms, which we wish to cancel. The cancellation of the unwanted terms yields M conditions (BAEs) for the parameters λ_α. The eigenvalue for the transfer matrix of the HC is

$$\Lambda_{\mathrm{HC}}(u) = a^L(u)\prod_{\alpha=1}^{M} a^{-1}(u-\lambda_\alpha+\mathrm{i}/2) + \prod_{\alpha=1}^{M} a^{-1}(\lambda_\alpha - u - \mathrm{i}/2). \tag{6}$$

The BAEs for rapidities have the form

$$a^L(-\lambda_\alpha - \mathrm{i}/2) = \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{M} \frac{a(\lambda_\beta - \lambda_\alpha)}{a(\lambda_\alpha - \lambda_\beta)} \qquad j = 1, \ldots, M. \tag{7}$$

The energy of the Hamiltonian ($\mathcal{H}_{\mathrm{HC}}$) (determined as the logarithmic derivative of the eigenvalue of the transfer matrix) and the magnetization have the simple form

$$\begin{aligned} E_{\mathrm{HC}} &= -\pi J \sum_{j=1}^{M} a_1(\lambda_\alpha) + E_F \\ M^z_{\mathrm{HC}} &= \frac{L}{2} - M \end{aligned} \tag{8}$$

where E_F is the energy of the ferromagnetic state with all spins up and $a_n(x) = 2n/\pi(4x^2+n^2)$. One can easily include an external magnetic field H in the consideration, too: $E_{\mathrm{HC}}(H) = E_{\mathrm{HC}}(H=0) - HM^z_{\mathrm{HC}}$. The BAE can be solved for any M. The method for the solution of the BAE in the *thermodynamic limit* (L, $M \to \infty$, M/L being fixed) was suggested by Hulthén [5]. In this limit one deals with the *distributions* of rapidities $\rho(\lambda) = \lim_{L\to\infty}[L(\lambda_{\alpha+1} - \lambda_\alpha)]^{-1}$ (which satisfy Fredholm integral equations). The thermodynamics of the HC can be constructed either using the 'string hypothesis' [6] or with the help of the so-called *thermal* Bethe ansatz [7, 8].

The (relative) simplicity of the Bethe ansatz scheme for the HC (e.g., in its algebraic realization, the QISM) has attracted various studies with a number of very important generalizations. First, the exact results for the $SU(2)$-symmetric spin-$\frac{1}{2}$ HC have been generalized for the cases of lower spin symmetry: the uniaxial $U(1)$-symmetric case [9] and the case of biaxial spin symmetry [1] (so-called XXZ and XYZ models, respectively). For these generalizations the coordinate (original) Bethe ansatz scheme was used first, but later it was understood that in the framework of the QISM one can obtain, for example, a $U(1)$-symmetric generalization using the algebraic quantum group ('q-deformation') approach (which serves for a construction of solutions to YBE; then the QISM can be used). Later, on the basis of the QISM, the solution for the $SU(2)$-symmetric higher-spin model (the so-called Takhtajian–Babujian model) [10, 11] was discovered. The other very important generalization of the HC for the higher-spin situation was the Uimin–Sutherland $SU(2S+1)$-symmetric model [12, 13] with the *nesting* BAE for additional, nesting rapidities (which describe additional degrees of freedom of $SU(2S+1)$ symmetry compared with $SU(2)$). Naturally, the latter cases were also exactly solved for the uniaxial and biaxial situations. Another important generalization of the two-dimensional solvable models was the introduction of inhomogeneous shifts of the spectral parameter [1]; later such shifts were considered in one-dimensional quantum field

theories (QFTs) [14]. The generalization of the HC model was also realized in a somewhat different direction: One can study the spin–spin pair coupling not between the NN spins but rather between pairs of spins connected by chords of a spin chain closed into a ring (so-called $1/r^2$ interactions) with the Haldane–Shastry model [15, 16] analogous to the HC with the NN interactions. Later it was shown (see, e.g., [17]) that Bethe ansatz solvable models with NN interactions and with $1/r^2$ interactions are in fact two limiting cases of more general construction of 'elliptic' couplings between pairs of particles.

Exactly solvable one-dimensional quantum spin models are also closely related to the exactly solvable one-dimensional models of correlated fermions. The well known Jordan–Wigner transformation maps a one-dimensional system of spins $\frac{1}{2}$ onto a one-dimensional system of spinless fermions [18]. On the other hand, a large class of $SU(n|m)$-(super)symmetric models of one-dimensional correlated fermionic systems was discovered (for the simplest case of electrons with spin of one kind $n = 1$ and $m = 2$ [19, 20]) and solved (in the framework of a *grading* scheme of the QISM; see, e.g., [21]). another seminal model of one-dimensional correlated electrons—the one-dimensional Hubbard model—was also exactly solved by the Bethe ansatz [22].

2. Integrable models with several different couplings between particles

As we showed in the introduction, the main feature of the Bethe ansatz solvable models is only elastic scattering between particles. This is mathematically formulated as the YBEs and, hence, the latter for two-particle scattering matrices (and, thus, for L-operators, R-matrices and monodromies) are the necessary conditions for any system to be Bethe ansatz integrable. This is why for the construction of any new exactly Bethe ansatz solvable Hamiltonians one needs to look for two-particle scattering matrices satisfying the YBE. In fact, the quantum group approach is one of the possible realizations of such a search.

Another example of such a program was used for studies of integrable systems with one particle being involved in interactions with not only two NNs (or infinitely many particles as in the case of $1/r^2$ models), but with four (as for the two-dimensional lattice), etc, i.e. with a *finite* number of other particles. This is the case for several coupled chains of quantum particles. Naturally, such systems belong to the intermediate class between quantum one-dimensional systems without any inter-chain interactions and two-dimensional systems (with an infinite number of coupled quantum chains). It is known that for Bethe ansatz integrable two-dimensional quantum systems (or associated three-dimensional statistical systems) one has to consider not the YBEs (also called 'triangular' equations) but rather 'tetrahedron' equations [23] (see also [24]). However, due to the huge number of equations to be solved in the framework of the latter construction, it was understood that it was necessary to look for some simplifications (i.e. to consider coupled-chain systems within the YBE but for some modified operators). Note that in fact these realizations are some simple projections of 'tetrahedron' equations onto the YBE.

Probably the first example of such a scheme was used by Shastry [25] when he mapped a one-dimensional Hubbard chain onto *two coupled spin chains* with special (Ising-like) interaction between them:

$$\begin{aligned}\mathcal{H}_H &= -t\sum_{j,\tau=\pm}(c^{\dagger}_{j+1,\tau}c_{j,\tau}+\text{H.c.})+U\sum_j(n_{j,+}-\tfrac{1}{2})(n_{j,-}-\tfrac{1}{2})\\ &\equiv -t\sum_{j,l=1,2}(\sigma^{+}_{l,j+1}\sigma^{-}_{l,j}+\text{H.c.})+U\sum_j\sigma^{z}_{1,j}\sigma^{z}_{2,j}\end{aligned} \tag{9}$$

where $c^{\dagger}_{j,\tau}$ creates an electron with spin τ at site j ($n_{j,\tau}=c^{\dagger}_{j,\tau}c_{j,\tau}$), and $\sigma^{\pm}_{l,j}=\sigma^{x}_{l,j}\pm \mathrm{i}\sigma^{y}_{l,j}$ and

$\sigma^z_{l,j}$ are operators of projections of spin situated at site j of the chain $l = 1, 2$. In fact in the QISM for the Hubbard model introduced in [25] $\check{L}$-operators (we will use the checked symbols for the two- and multi-chain models below) for these two spin chains were constructed as

$$\check{L}(u) = \exp(h\sigma^z_{1,a}\sigma^z_{2,a}) \times L_{1,j,a}(u) \times L_{2,j,a} \times \exp(h\sigma^z_{1,a}\sigma^z_{2,a}) \tag{10}$$

i.e. a product of two L-operators for each chain ($L_{l,j,a}(u)$, a denotes the auxiliary space), but with some exponential multiplier (coupling) between them (with the constraint $4t \sinh 4h = U \sinh 2u$). (Note that the exact Bethe ansatz solution to the one-dimensional Hubbard chain was already known [22]; this is why the work [25] was a new way of obtaining the known results but also allowed for a systematic construction of the integrals of motion, which was very important, in the framework of the QISM.) It was shown that the $\check{L}$-operator with such a construction satisfied the YBE, hence the whole scheme of the Bethe ansatz could be applied. Similar constructions with additional factors, which depend only on σ^z-operators, for $\check{L}$-operators were discovered for other (already new) classes of quantum multichain models with interactions between chains (or, in other words, for models with many kinds of particle) [26–30] (later re-visited [31]) and for related models with particles of anyon-like fractional exclusion statistics [32, 33]. However for all multi-chain (we shall use this interpretation below) models based on that simple construction of $\check{L}$-operator the numbers of excitations in each chain were conserved (i.e. excitations *could not move from one chain to another*). Excitations can propagate only *along* and not *across* chains. (In the case of the one-dimensional Hubbard chain this is the trivial consequence of the fact that the numbers of excitations with spin up and down are conserved separately). This is a very severe condition and seems to be a non-generic property of multi-chain quantum models.

2.1. Algebra of the basic integrable two-chain model

Another class of Bethe ansatz solutions to a multi-chain problem was discovered in [34–36] (later some of the results of [34–36] were reproduced in [37]). Namely, we considered the two-chain $\check{L}$-operator for the site j of the form

$$\check{L}_j(u) = L_{\sigma_{1,a},\sigma_{2,j}}(u+\theta) \otimes L_{\sigma_{1,a},\sigma_{1,j}}(u) \otimes L_{\sigma_{2,a},\sigma_{2,j}}(u) \otimes L_{\sigma_{2,a},\sigma_{1,j}}(u-\theta) \tag{11}$$

(where the index j denotes the quantum space, a denotes the auxiliary space) of the product of four L-operators (with the standard structure of an L-operator of a single Bethe ansatz solvable chain) each of which satisfies the YBE. The parameter θ determines the strength of coupling between the chains. One can easily check that $\check{L}$-operators satisfy the YBE, too. The structure of the $\check{L}$-operator is very clear: we have *two* chains, hence we have to consider *two quantum spaces and two auxiliary spaces* (one set for each chain). The $\check{R}$-operator for the two-chain problem can be constructed similarly to a single-chain case. The normalization for $\check{R}$-matrices implies $\check{R}(u)\hat{P}\check{R}(-u)\hat{P} = 1$. The monodromy matrix for this two-chain problem is the ordered product of L $\check{L}$-operators

$$\check{T}_L(u,\theta) = \check{L}_L(u,\theta)\check{L}_{L-1}(u,\theta)\cdots\check{L}_1(u,\theta). \tag{12}$$

Naturally, this monodromy manifests the result of scatterings of *two* auxiliary particles off *two sets* of physical particles in each of the chains. Such a monodromy operator satisfies the YBE by construction. The transfer matrix $\check{\tau}(u,\theta) = \mathrm{Tr}[\check{T}_L(u,\theta)]$ is the trace (over two auxiliary spaces) of the monodromy. The associated two-dimensional statistical vertex problem is, naturally, the *two-layered* one. Clearly, transfer matrices with different u and θ commute (with θ being fixed). Hence $\check{\tau}(u,\theta)$ can be used as a generating functional for an infinite number of conservation laws, and, this is why the problem is exactly solvable. It is interesting to point out [35, 36] that the transfer matrix of the two-chain problem can be written

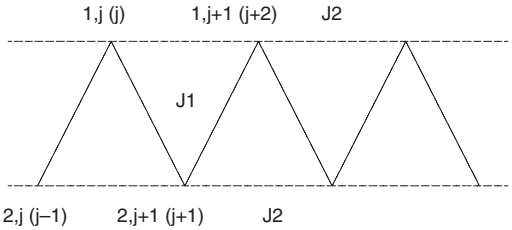

Figure 1. The two-chain spin-$\frac{1}{2}$ model with NN ($J1 = \frac{2J}{1+\theta^2}$) and NNN ($J2 = \frac{\theta^2}{1+\theta^2}$) interactions. Note that lattice sites are notated in two equivalent ways: in a two-chain enumeration and in a single-chain one (see the text).

as $\check{\tau}(u,\theta) = \tau(u) \otimes \tau(u+\theta)$. The Hamiltonian for the two-chain problem can be determined as the logarithmic derivative of the transfer matrix $\check{\tau}(u,\theta)$ taken at $u = 0$. The Hamiltonian of two coupled Heisenberg chains of spins $\frac{1}{2}$ can be written as

$$\mathcal{H}_2 = \frac{J}{1+\theta^2}\sum_j \theta^2(\boldsymbol{S}_{1,j}\boldsymbol{S}_{1,j+1} + \boldsymbol{S}_{2,j}\boldsymbol{S}_{2,j+1}) + 2\boldsymbol{S}_{1,j}(\boldsymbol{S}_{2,j} + \boldsymbol{S}_{2,j+1}) \\ +2\theta(\boldsymbol{S}_{1,j} - \boldsymbol{S}_{2,j+1})[\boldsymbol{S}_{1,j+1} \times \boldsymbol{S}_{2,j}] \tag{13}$$

where $[. \times .]$ denotes a vector product. This is the Hamiltonian of the two-chain spin model with a zigzag interchain coupling (see figure 1 for an illustration). In fact one can re-write equation (13), i.e. to re-numerate spins. It is easy to see that after such a re-numeration the Hamiltonian $\mathcal{H}_2$ is the Hamiltonian of a single spin chain of length $2L$ with NN and next-nearest-neighbour (NNN) interactions

$$\mathcal{H}_2 \equiv \mathcal{H}_{\rm NNN} = \frac{J}{1+\theta^2}\sum_j 2\boldsymbol{S}_j\boldsymbol{S}_{j+1} + \theta^2\boldsymbol{S}_j\boldsymbol{S}_{j+2} - 2\theta(\boldsymbol{S}_{j-1} - \boldsymbol{S}_{j+2})[\boldsymbol{S}_j \times \boldsymbol{S}_{j+1}]. \tag{14}$$

For the two-chain problem one can proceed along the same lines of the QISM as for a single-chain one. We obtain for the eigenvalue for the transfer matrix

$$\Lambda_2(u) = a^L(u)a^L(u+\theta)\prod_{\alpha=1}^{M} a^{-1}(u - \lambda_\alpha + \mathrm{i}/2) + \prod_{\alpha=1}^{M} a^{-1}(\lambda_\alpha - u - \mathrm{i}/2). \tag{15}$$

The BAEs for rapidities have the form

$$a^L(-\lambda_\alpha - \mathrm{i}/2)a^L(-\lambda_\alpha - \theta - \mathrm{i}/2) = \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{M} \frac{a(\lambda_\beta - \lambda_\alpha)}{a(\lambda_\alpha - \lambda_\beta)} \qquad j = 1, \ldots, M. \tag{16}$$

The energy of the Hamiltonian ($\mathcal{H}_2 = \mathcal{H}_{\rm NNN}$) can be written as (note that the energy is the logarithmic derivative of the eigenvalue of the transfer matrix (15))

$$E_2 = -\pi J\sum_{j=1}^{M}(a_1(\lambda_\alpha) + a_1(\lambda_\alpha + \theta)) + E_F \tag{17}$$

where E_F is again the energy of the ferromagnetic state with all spins up and the magnetization is $M_2^z = 2L(1/2) - M = L - M$. Clearly excitations can move from one chain to another (in contrast to the previously considered class of multichain quantum models).

One can easily check that the Hamiltonian and BAE reproduce two limiting cases [35,36]: for $\theta = 0$ one deals with a single HC of length $2L$, while the case $\theta \to \infty$ corresponds to two disconnected single HCs of lengths L.

Let us consider the structure of the Hamiltonian equation (13) or (14). It turns out that the last (three-spin) term can be written in other ways, namely [34, 35, 51]

$$\mathcal{H}_3 = \frac{2J\theta}{\theta^2+1}\sum_j (-1)^j \epsilon^{klm} S_j^k S_{j+1}^l S_{j+2}^m$$
$$= \frac{2J\mathrm{i}\theta}{\theta^2+1}\sum_j [\boldsymbol{S}_{1,j}(\boldsymbol{S}_{2,j}+\boldsymbol{S}_{2,j+1}), (\boldsymbol{S}_{1,j}\boldsymbol{S}_{1,j+1}+\boldsymbol{S}_{2,j}\boldsymbol{S}_{2,j+1})] \tag{18}$$

where [., .] denotes a commutator. Obviously this term breaks time-reversal (T) and parity (P) symmetries separately; however, TP is conserved. In other words, only the *common* replacement of $1 \leftrightarrow 2$ and $j+1 \leftrightarrow j-1$, etc, leaves the Hamiltonian unchanged. Physically the origin of such an interaction may be due to the spin–orbit coupling in the case where the orbital motion of electrons is frozen at sufficiently low temperatures [35]. In fact one can determine a local *spin chirality* as $\epsilon^{klm} S_1^k S_2^l S_3^m$ (where 1, 2, 3 numerate the NNs) (see, e.g., [34, 35, 38] for multi-chain systems and [39, 40] for two-dimensional spin lattices). Hence for real θ the three-spin term can be considered as a *spin chirality field* in the two-chain Hamiltonian [34–36, 51] (in fact, it is the *staggered* field, thus one can speak about the spin *antichirality* [34, 51]). In other words, this value measures *spin currents* around each of the elementary plaquettes. In the classical limit the effect of this term is analogous to (in a phase with a zero ferromagnetic moment) the time component of some topological (Noether) current [35, 36]

$$I_0 = \frac{\theta}{2\pi}\int \mathrm{d}^2x\, \epsilon^{klm}\epsilon_{\mu\nu} n^k \partial_\mu n^l \partial_\nu n^m \qquad \mu, \nu = 1, 2 \tag{19}$$

where n^i ($i = x, y, z$) is the vector of the classical density of spin with $\boldsymbol{n}^2 = 1$. This three-spin term is the definition of the Pontryagin index (topological charge or winding number [41]), which determines topologically nontrivial distributions of the spin density in the system. In (3 + 1) systems one has to study Hopf invariants [42]; for the (2 + 1) ones the same role is played by the topological Chern–Simons term [44], while in the (1 + 1) system it is the so-called θ-vacuum [45] (see, also, [46, 47]). (It turns out that for the (1 + 1) and (3 + 1) cases it is impossible to construct a conserved scalar or pseudoscalar with the properties of the topological charge: for example, for the (3 + 1) situation one has a vector I_α instead of a scalar I_0.) For the (2 + 1) case I_0 is the characteristic of the homotopy class $\pi^2(S^2) = Z$— the distribution of $\boldsymbol{n}$ is topologically nontrivial if $\pi^2(S^2) \neq 0$ [43]. From this viewpoint the three-spin term is a remarkable example of the two-dimensional nature of the system seen even for two coupled chains (definitely such terms should be absent from the pure one-dimensional case, with only NN interactions, cf [46]). In the classical limit this term is the total time derivative $\partial_0 I_0 = 0$ [35, 36], hence it does not affect classical equations of motion (and classical configuration of spins [38]). However, it is very important in the quantum mechanics [35, 36, 38, 51], see below. Namely, the presence of this term destroys the spin gap of low-lying excitations [35, 38], and this is why in the long-wave limit the model can be described as a conformal field theory. The effect of a magnetic field perpendicular to the plane passing through the chains is also analogous to our topological three-spin term [35, 48]: this term is an *effective surface term* in the spin space. Models of this class of exactly solvable lattice multi-chain quantum spin models are a good example of *chiral spin liquids* [39] but with explicitly (not spontaneously) broken T and P symmetries.

For real θ (and for positive J) the first two terms in the Hamiltonian equations (13) and (14) correspond to the antiferromagnetic NN and NNN interactions, thus to the *spin-frustrated* case. On the other hand, the *imaginary* values of θ pertain to the non-Hermitian Hamiltonian (three-spin term) and either AF or ferromagnetic (F) NN and NNN interactions.

It is clear that in fact the Hamiltonian equation (13) consists of the sum of two *commuting* parts $\mathcal{H}_2 = \bar{\mathcal{H}}_2 + \tilde{\mathcal{H}}_2$. The first part, $\bar{\mathcal{H}}_2$, includes pair interactions only between the spins of chain 1 and between the spins of chains 1 and 2, and three-spin couplings between two spins from chain 1 and one spin from chain 2, while $\tilde{\mathcal{H}}_2$ includes the remnant couplings. For each of these parts we have the $\tilde{L}$-operator being the direct product of *only two* L-operators of a single HC (each with their own auxiliary spaces and *alternating* shifts of *the same* spectral parameter). Naturally, their transfer matrices commute. The eigenvalues for the transfer matrices for each of these subsystems have the form of equation (15). The Hamiltonians of each of these parts have also a 'triangular' structure. In the language of a single chain, equation (14), the latter decomposition means that one can decouple the total system into two subsystems, for each of which one has pair spin NN interactions and NNN interactions only between even (odd) sites. Obviously, one can consider these two subsystems with different overall constants $J_{1,2}$ and of different lengths $L_\pm$ [49, 50], while the shift θ has to be the same for both subsystems (to satisfy the YBE). In the next sections we shall deal with the simplest (and, probably, the most important) situation $J_1 = J_2 = J > 0$ and $L_+ = L_- = L$, and return to the case with different exchange constants and lengths in the following sections.

2.2. The nonmagnetic ground state

After a simple change of variables, equations (16) and (17) can be written as

$$\begin{aligned} &\prod_{\pm} e_1^L(\lambda_\alpha \pm \theta/2) = \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{M} e_2(\lambda_\alpha - \lambda_\beta) \qquad \alpha = 1, \ldots, M \\ &E_2 - E_F = -H(L-M) - \pi J \sum_{\pm} \sum_{\alpha=1}^{M} a_1(\lambda_\alpha \pm \theta/2) \end{aligned} \tag{20}$$

where $e_n(x) = (2x + \mathrm{i}n)/(2x - \mathrm{i}n)$. Note that $a_n(x) = (\mathrm{i}/2\pi)(\partial e_n(x)/\partial x)$. Let us first concentrate on real values of θ, i.e. on the Hermitian Hamiltonians. It is easy to show that in the ground state the solutions of the BAE pertain to only real values of rapidities λ_α [34–37]. The solution to the BAEs (20) is usually obtained in the thermodynamic limit: Instead of the discret set of rapidities one introduces a distribution of a continuous density of rapidities. The ground state pertains to solutions of the BAE with negative energies, i.e. it is connected with the filling up of Dirac sea(s) for a model. In the thermodynamic limit the real roots of equations (20) are continuously distributed over certain intervals, which determine the Dirac sea(s) of the two-chain model. The set of integral equations for the 'dressed' densities of rapidities λ_α ($\rho(\lambda)$) and 'dressed' energies of low-lying excitations ($\varepsilon(\lambda)$) are

$$\rho(\lambda) + \int_{(Q)} \mathrm{d}\nu\, K(\lambda - \nu)\rho(\nu) = \sum_{\pm} \rho_\pm^0(\lambda \pm \theta/2) \tag{21}$$

and

$$\varepsilon(\lambda) + \int_{(Q)} \mathrm{d}\nu\, K(\lambda - \nu)\varepsilon(\nu) = \sum_{\pm} \varepsilon_\pm^0 \tag{22}$$

where the kernel of integral equations is $K(u) = a_2(u)$. The values

$$\rho_\pm^0(u) = a_1(u \pm \theta/2) \equiv \mathrm{i}\frac{\mathrm{d}p_\pm^0(u \pm \theta/2)}{\mathrm{d}u} \tag{23}$$

are 'bare' densities of rapidities ($p = \sum_\pm p_\pm^0(\lambda \pm \theta/2) + \pi$ is the total 'bare' momentum of the system) and

$$\varepsilon_\pm^0(\lambda) = \frac{H}{2} - \pi J a_1(\lambda \pm \theta/2) \tag{24}$$

are 'bare' energies (here 'bare' corresponds to the case without interactions, i.e. without integral terms, and an interaction 'dresses' them as usual). The integrations are performed over the domain (Q), determined in such a way that the dressed energies inside these intervals are negative. The limits of integrations are determined by zeros of the dressed energies, and are the *Fermi points* for each sea.

For zero magnetic field ($M_2^z = 0$) it is easy to solve the problem (in this case all λ are distributed along the real axis from $-\infty$ to ∞) and the ground-state energy is [34–38]

$$E_2 - E_F = -2LJ \int_{-\infty}^{\infty} d\omega \frac{\cos^2(\omega\theta/2)}{1 + \exp|\omega|}$$
$$= -2LJ \ln 2 - \frac{LJ}{2} \sum_{\pm} \left[\Psi\left(\frac{2 \pm i\theta}{2}\right) - \Psi\left(\frac{1 \pm i\theta}{2}\right) \right] \tag{25}$$

where $\Psi(x)$ is a digamma function and $E_F = LJ(2 + \theta^2)/4(1 + \theta^2)$. It turns out that the ground-state energy is an even function of the coupling parameter θ. Obviously the ground-state value of $E_2 - E_F$ is also a periodic function of θ: it is not changed by the replacement $\theta \to \theta + 2in$ (n integer). This property is connected with the topological effects of the $\mathcal{H}_3$ term in the Hamiltonian [35].

The lowest excitation is a *spinon* [37, 38, 51, 64] (a hole in the distribution of rapidities λ [52, 53]). Note that, as usual for periodic boundary conditions, for topological reasons these spinons can appear only in pairs. Suppose we have a hole with the rapidity λ_0. Than the quasimomentum of a spinon is

$$p(\lambda_0) = 2 \tan^{-1}\left(\frac{\sinh(\pi\lambda)}{\cosh(\pi\theta)}\right) \tag{26}$$

while its energy is

$$\varepsilon_{sp} = -J \frac{\partial p}{\partial \lambda_0} = \frac{\pi J}{2} \sum_{\pm} \frac{1}{\cosh(\lambda_0 \pm \theta/2)}. \tag{27}$$

It is easy to find the dispersion law for a spinon (see figure 2) [51]

$$\varepsilon_{sp}(p) = \pi J \sin\left(\frac{p}{2}\right) \sqrt{1 - \tanh^2\left(\frac{\pi\theta}{2}\right) \sin^2\left(\frac{p}{2}\right)} \tag{28}$$

where p is the quasimomentum of the spinon. The spinon is gapless. One can see that $\varepsilon_{sp}(\theta) = \varepsilon_{sp}(\theta + 2in)$ (n is integer), which also pertains to the topological properties of the $\mathcal{H}_3$ term in the Hamiltonian [38]. Hence elementary excitations of the two-chain model carry nonzero topological charge (measured by n). It turns out that the dynamics of the magnetization (near $p = 0$) is connected with the gapless part of the spinon's dispersion law, while the dynamics of the staggered magnetization (near $p = \pi$) pertains to the gapped part.

Let us consider conformal properties of the system. As a consequence of the conformal invariance of (1 + 1) quantum systems, the classification of universality classes is simple in terms of the central charge (conformal anomaly c) of the underlying Virasoro algebra [54–57]. Critical exponents in a conformally invariant theory are scaling dimensions of the operators within the quantum model. They can be calculated considering the finite-size (mesoscopic) corrections to energies and quasimomenta of the ground state and low-lying excited states. Conformal invariance formally requires all gapless excitations to have the same velocity (Lorentz invariance). The complete critical theory for systems with several gapless excitations with *different* Fermi velocities is usually given as a *semidirect product* of these independent Virasoro algebras [58–62]. For $H = 0$ the conformal limit of the two-chain model corresponds to a level-one Kac–Moody algebra (one Wess–Zumino–Novikov–Witten (WZNW) model

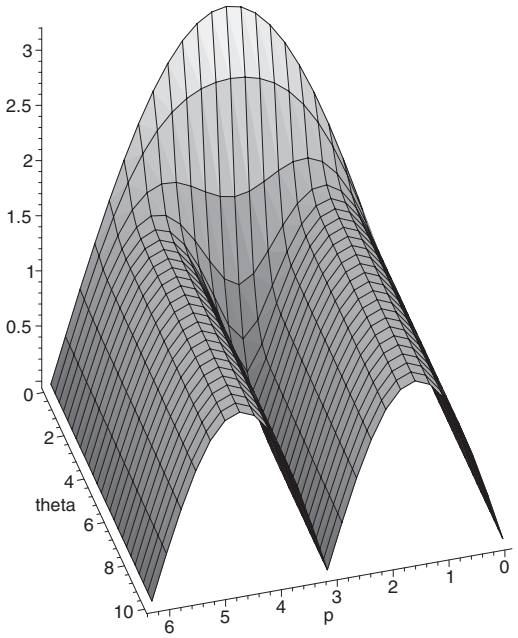

Figure 2. The dispersion law $\varepsilon_{\rm sp}(p)$ for the 'basic' two-chain spin $\frac{1}{2}$ model as a function of the quasimomentum p and coupling θ. The exchange constant J is equal to unity; $H = 0$.

of level one with the conformal anomaly $c = 1$). The finite-size correction to the energy ($E = E_2 + E_F + E_{\rm fs} + \cdots$) is rather standard [63]

$$E_{\rm fs} = -\frac{\pi v}{12L} + \frac{\pi v}{L}(\Delta_{\rm l} + \Delta_{\rm r}) \tag{29}$$

where $v = \pi J/2$ is the Fermi velocity of the dispersion law, linearized about the Fermi points ($p = 0, 2\pi$), of a spinon, and the conformal dimensions of primary operators are (indices denote the conformal dimensions for right- and left-moving quasiparticles, at the right and left Fermi point, respectively)

$$2\Delta_{\rm l,r} = \left(\frac{\Delta M}{2z} \pm z\Delta D\right)^2 + 2n_{\rm l,r}. \tag{30}$$

Here ΔM is an integer denoting the change of the number of particles induced by the primary operator, ΔD is an integer (half-integer) denoting the number of transferred particles from the right to the left Fermi point (backward scattering processes) and $n_{\rm l,r}$ are the numbers of particle–hole excitations of right- and left-movers. The values for the quantum numbers are restricted by the condition $\Delta D = \Delta M/2$ (mod 1). The 'dressed' charge $z = \xi(Q)$ (one can see that this quantity measures the number of bare particles which form the dressed excitation $\xi = -\partial\varepsilon/\partial\mu$, where μ is the generalized chemical potential) is the solution of the (standard) integral equation [58–60]

$$\xi(\lambda) + \int_{(Q)} {\rm d}\nu\, K(\lambda - \nu)\xi(\nu) = 1 \tag{31}$$

taken at the limits of integration $\lambda = \pm\infty$ (these are the Fermi points, symmetric with respect to zero). The dressed charge is a scalar, and z is equal to $1/\sqrt{2}$ (recall, for zero magnetization). Correlation functions of low-lying excitations decay asymptotically $\propto (x - {\rm i}vt)^{-\Delta_{\rm l}}(x + {\rm i}vt)^{-\Delta_{\rm r}}$.

The choice of the appropriate quantum numbers of excitations ΔM, ΔD and $n_{1,r}$ is determined for the leading asymptotics of correlators by taking possible numbers with the smallest exponents. Note that in fact nothing depends on the coupling constant θ in the conformal limit for $H = 0$, i.e. the conformal behaviour of the two-chain model at $H = 0$ is the same as for an HC [63].

2.3. Ground-state properties in an external magnetic field

Let us now switch on an external field $H \neq 0$. Here the situation appears to be very different for different values of the coupling constant θ [38, 64].

For small $\theta < \theta_c = 1/\sqrt{3}$ there is only one Fermi sea for spinons (i.e. there is only one minimum in the distribution of dressed energies). This is in a one-to-one correspondence with the behaviour of the dispersion law of a spinon: the latter has only one extremum (maximum) at $p = \pi/2$ for any values of $H \leqslant H_s$ ($H_s = 4J/(1+\theta^2)$). For $H \geqslant H_s$ (where $M = 0$) the system is in the *spin-saturation*, ferromagnetic (F) phase at zero temperature and spinons become gapped. The magnetic susceptibility manifests a square root singularity at H_s: $\chi(H) = 2/\pi\sqrt{(H_s - 3J)(H_s - H)}$. It is nothing other than the one-dimensional van Hove singularity of an empty Dirac sea of spinons. For $H \leqslant H_s$ the conformal behaviour can be described by equations (29)–(31) with $Q = Q(H)$ as for the $H = 0$ case (the only difference is that now the Fermi velocity of the spinons depends on H and z varies between $1/\sqrt{2}$ at $H = 0$ and 1 for $H \to H_s$). In a weak magnetic field the magnetic susceptibility is given by $\chi = (2\pi v)^{-1} = (\pi^2 J)^{-1}(1 + (2|\ln AH|)^{-1} - (\ln|\ln AH|/4\ln^2 AH) + \cdots)$ where A is a non-universal constant [34, 36] (logarithmic corrections appear due to the $SU(2)$-symmetry of the model).

At $\theta = \theta_c$ there is also one minimum in the distribution of dressed energies and, hence, only one maximum of the dispersion law of a spinon. However, these extrema are flatter: instead of the behaviour of the dressed energies $\sim \lambda^2$ for $\theta < \theta_c$, for $\theta = \theta_c$ it is $\sim\lambda^4$. This is why the magnetic susceptibility manifests the different singularity $\chi(H) = 1/3J\pi(H_s - H)^{3/4}$ at H_s (which is equal to $3J$ at θ_c). For $H \leqslant H_s$ the conformal behaviour is similar to the previous case, while for $H \geqslant H_s$ elementary excitations are qapped.

For $\theta > \theta_c$ there are two minima and one maximum in the distribution of dressed energies (the dispersion law of a spinon also reveals one minimum and two maxima) (see figure 3 for an illustration). Hence, the ground-state behaviour strongly depends on the value of external magnetic field. There are *two* critical values of the field for $\theta > \theta_c$. For $H \geqslant H_s = J[1 + (\sqrt{1+\theta^2}/\theta)]$ the two-chain system is in the spin-saturated phase with gapped spinon excitations. The magnetic susceptibilty also manifests square-root singularity at H_s ($\chi \sim 1/\sqrt{H_s - H}$). For $H < H_s$ spinons are gapless. There exists an *additional critical value* of external magnetic field, H_c [38, 64] (its dependence on θ can be approximated by $H_c \approx \pi J/\cosh(\pi\theta/2)$ [38]). For $H < H_c$ there is only *one* Dirac sea for spinons. The conformal behaviour is similar to the previous cases, i.e. we have *one* WZNW model of level one with the conformal anomaly $c = 1$. However for $H > H_c$ the behaviour is drastically changed: there are *two* Dirac seas for spinons (with *four* Fermi points) [38, 64]. In fact one can speak about the onset of the Dirac sea of 'holes' of spinons for $H > H_c$ [38]: according to this picture the critical field H_c pertains to the van Hove singularity of the empty band of these 'holes' of spinons. Note that at $\theta = \theta_c$, $H_c = H_s$, i.e. this point is *tricritical*. For $\theta > \theta_c$, $H > H_c$, the conformal limit of the two-chain model corresponds to the semidirect product of *two* level-one Kac–Moody algebras, both with conformal anomalies $c = 1$, i.e. to *two* WZNW models both of level one [38, 64]. The Dirac seas (i.e. possible spinons with negative energies) are in the intervals $[-Q^+, -Q^-]$ and $[Q^-, Q^+]$ (minima in the distributions of rapidities at

$\mp\theta/2$). This can be interpreted as the symmetrically distributed (around zero) Dirac seas of 'particles' for $[-Q^+, Q^+]$ and the Dirac sea of 'holes' for $[-Q^-, Q^-]$. Naturally, the Fermi velocities of 'particles' are positive, $v^+ = (2\pi\rho(Q^+))^{-1}\varepsilon'(\lambda)|_{\lambda=Q^+}$, while the Fermi velocities of 'holes' are negative, $v^- = -(2\pi\rho(Q^-))^{-1}\varepsilon'(\lambda)|_{\lambda=Q^-}$. The finite-size corrections to the energy for this case are

$$E_{\rm fs} = -\frac{\pi}{12L}(v^+ + v^-) + \frac{\pi}{L}\big(v^+(\Delta_{\rm l}^+ + \Delta_{\rm r}^+) + v^-(\Delta_{\rm l}^- + \Delta_{\rm r}^-)\big) \tag{32}$$

where the dispersion laws of 'particles' and 'holes' are linearized about the Fermi points for each Dirac sea. The conformal dimensions of primary operators are (the upper indices denote Dirac seas; the lower indices denote right and left Fermi points of each of these two Dirac seas):

$$2\Delta_{\rm l,r}^{\mp} = \left[\frac{(x_{-\pm}\Delta M^+ - x_{+\pm}\Delta M^-)}{2\det\hat{x}} \mp \frac{(z_{-\pm}\Delta D^+ - z_{+\pm}\Delta D^-)}{2\det\hat{z}}\right]^2 + 2n_{\rm l,r}^{\mp} \tag{33}$$

where the 'minus' sign between the terms in square brackets corresponds to the right-movers and 'plus' to the left-movers. Here $\Delta M^{\pm}$ denote the differences between the numbers of particles excited in the Dirac seas of 'particles' and 'holes', labelled by the upper indices. $\Delta D^{\pm}$ denote the numbers of backward-scattering excitations, and $n_{\rm l,r}^{\pm}$ are the numbers of particle–hole excitations for right- and left-movers of each of the Dirac seas (for 'particles' and 'holes'). Note that $\Delta M^{\pm}$ and $\Delta D^{\pm}$ are *not independent*. Their values are restricted by the following relations: $\Delta M^+ - \Delta M^- = \Delta M$ and $\Delta D^+ - \Delta D^- = \Delta D$, where ΔM and ΔD determine in a standard way the changes of the total magnetization and the total momentum of the system, respectively, due to excitations. *Four* backscattering low-lying excitations are possible; however, only *two of them are really independent*, due to the above-mentioned relations. The same is true for excitations that change the total magnetization of the system: there are only *two independent* of four such possible excitations. This is the direct consequence of the fact that *only one magnetic field* determines the filling of the Dirac seas for 'particles' and 'holes', or, in other words, the filling of the two Dirac seas for spinons centred at $\pm\theta/2$.

The dressed charges $x_{ik}(Q^k)$ and $z_{ik}(Q^k)$ $(i, k = +, -)$ are matrices in this phase. They can be expressed by using the solution of the integral equation [50, 65]

$$f(u|Q^{\pm}) = \left(\int_{-Q^+}^{Q^+} - \int_{-Q^-}^{Q^-}\right)K(u-v)f(v|Q^{\pm}) = K(u-Q^{\pm}) \tag{34}$$

with

$$\begin{aligned} z_{ik}(Q^k) &= \delta_{i,k} + (-1)^k\tfrac{1}{2}\left(\int_{Q^i}^{\infty} - \int_{-\infty}^{-Q^i}\right)\mathrm{d}v\, f(v|Q^k) \\ x_{ik}(Q^k) &= \delta_{i,k} - (-1)^k\int_{-Q^i}^{Q^i}\mathrm{d}v\, f(v|Q^k). \end{aligned} \tag{35}$$

Note that the dressed charges depend on the value of the coupling constant θ indirectly, only via the limits of integrations. In the first-order approximation one can write the solutions as $x_{ik}(Q^k) \approx \delta_{i,k} - (-)^k\int_{-Q^i}^{Q^i}\mathrm{d}v\,K(v-Q^k) + \cdots$ and $z_{ik}(Q^k) \approx \delta_{i,k} + (-)^k(1/2)(\int_{Q^i}^{\infty} - \int_{-\infty}^{-Q^i})\,\mathrm{d}v\,K(u-Q^k) + \cdots$. The Dirac sea for 'holes' disappears, naturally, for $H \to H_{\rm c}$, $\theta \to \theta_{\rm c}$. The slopes of the dressed energies of 'particles' and 'holes' at Fermi points of the Dirac seas (Fermi velocities) differ in general from each other. Therefore we have a semidirect product of two algebras. Hence, in this region the dressed charges are 2×2 matrices. At the critical line $H_{\rm c}$ the Dirac sea of 'holes' disappears (as well as the components of the dressed charge matrix $\hat{x}$) with square root singularities of the critical exponents for the

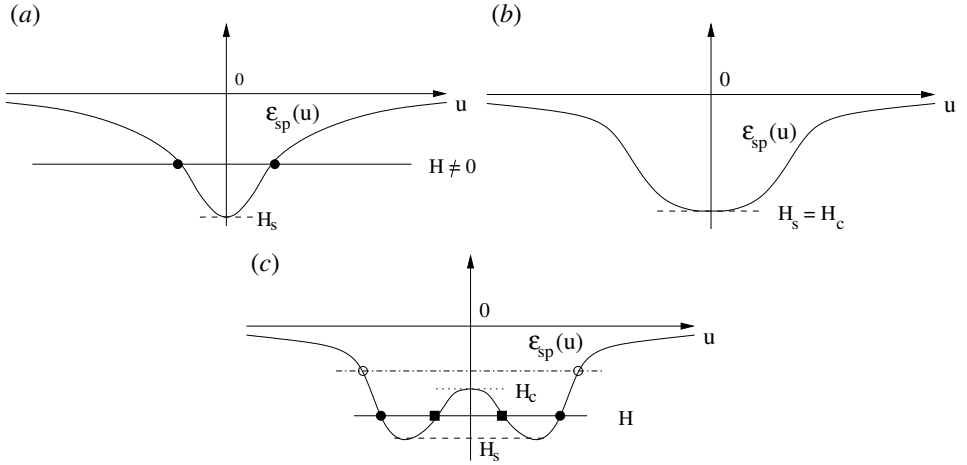

Figure 3. The qualitative picture for the dressed energy of spinons of the 'basic' two-chain model ε_{sp} as a function of the rapidity (u) for different coupling parameters θ, $\theta < \theta_c$ (a), $\theta = \theta_c$ (b) and $\theta > \theta_c$ (c). Depending on the value of the magnetic field H there is one Fermi sea in case (a), but up to two Fermi seas in case (c). The limiting case of a dressed energy function $\varepsilon_{sp}(u)$ with just one minimum and that with two minima and one maximum is shown in (b). Here all derivatives of order one to three vanish at $u = 0$. Filled circles and filled squares denote Fermi points for 'particles' and 'holes' for $H \neq 0$. Note that for the case (c) for $H < H_c$ there are only two Fermi points (open circles).

correlation functions [65]. Note that the dressed charge z becomes $z = (2x)^{-1}$ at the phase transition line H_c. This corresponds to the disappearance of one of the WZNW conformal field theories. Correlation functions of the two-chain model decay algebraically in this phase $\propto (x - iv^+t)^{-\Delta_l^+}(x + iv^-t)^{-\Delta_l^-}(x - iv^+t)^{-\Delta_r^+}(x + iv^-t)^{-\Delta_r^-}$ with minimal exponents of possible quantum numbers of excitations $\Delta M^{\pm}$, $\Delta D^{\pm}$ and $n_{l,r}^{\pm}$. We point out once again that the same magnetic field plays the role of a generalized chemical potential for the 'particles' and 'holes' (or, in other words, for spinons of both Dirac seas in the second phase), and hence this choice of 'minimal quantum numbers' is constrained.

We must point out here that there is a crucial difference between the two-chain model and the case of dressed charge matrices appearing for systems with the internal structure of bare particles [61, 62]. In the latter the two Dirac seas of the ground states are connected with different kinds of excitation, for example holons and spinons for the repulsive Hubbard model, or Cooper-like singlet pairs and spinons for the supersymmetric t–J model. They pertain to two different kinds of Lagrange multiplier (generalized chemical potentials): the chemical potentials and magnetic fields. Thus the low-lying excitations of the conformal theories in the spin and charge sectors of these correlated electron models are practically *independent* of each other (spin–charge separation). (Note that the spin and charge sectors are connected via the off-diagonal elements of the dressed charge matrix though. This is the consequence of the fact that, say, holons or unbound electrons carry *both charge and spin*.) On the other hand, two Dirac seas appear for the *same kinds of particle* for the two-chain models reviewed in this article, which are also connected with the *same* magnetic field, governing the filling of both Dirac seas. The latter appear due to two minima in the bare energy distribution and correspond to nonzero shift θ in the BAE. In other words, two Dirac seas are determined by the intra-chain coupling (in a two-chain picture) or by the NNN coupling (in a single-chain description) and appear if the values of coupling and external magnetic field are higher than the *threshold* values

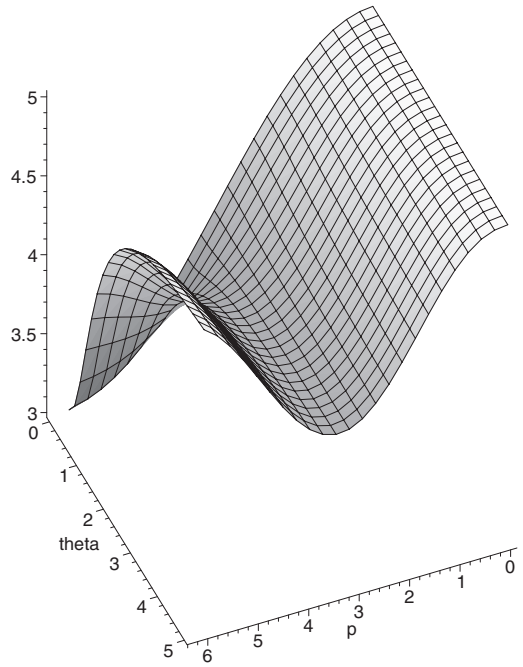

Figure 4. The dispersion law $\varepsilon_{\rm sp}(p)$ for the 'basic' two-chain spin-$\frac{1}{2}$ model as a function of the coupling θ and quasimomentum p. The exchange constant J is equal to unity; $H = 5$.

$\theta_{\rm c}$ and $H_{\rm c}$, respectively. We believe that such a threshold behaviour does not depend on the integrability of the model and is the generic feature for any multi-chain quantum spin model.

It is worthwhile to write down the dispersion for a spinon for $H > H_{\rm s}$ (see figure 4):

$$\varepsilon_{\rm sp}(p) = H - J\sin^2\left(\frac{p}{2}\right)\left[1 - \frac{\cos\left(\frac{p}{2}\right)}{\cos\left(\frac{p}{2}\right) + \sqrt{1+\theta^2\sin^2\left(\frac{p}{2}\right)}}\right]. \tag{36}$$

Note that instead of the cusp of figure 2 for large θ, the dispersion law of a spinon in an F state manifests a minimum.

The zero-temperature phase diagram (θ–H) is presented in figure 5. There are three phases: the spin-saturated (ferromagnetic) phase (with gapped excitations) and two phases with gapless excitations—the commensurate one for small θ and incommensurate [38, 50] for large θ and $H_{\rm c} < H < H_{\rm s}$.

2.4. Thermodynamics

The thermodynamics of the two-chain model in the framework of the 'string hypothesis' was built in [36] (see also [49] for the case with $J_1 \neq J_2$). However, some concerns appear about the validity of string solutions for large lengths of strings [66–69]. Therefore in this review article we shall use a different approach, the *thermal* Bethe ansatz [8] in the framework of the 'quantum transfer matrix' (QTM).

For our quantum two-chain spin Hamiltonian at finite temperature we found a suitable lattice path integral representation by a mapping preserving integrability. For a general formulation of Trotter–Suzuki decompositions the reader is referred to [70]. Let $R^{\mu_i\mu_{i+1}}_{\alpha_i\beta_i}(x)$

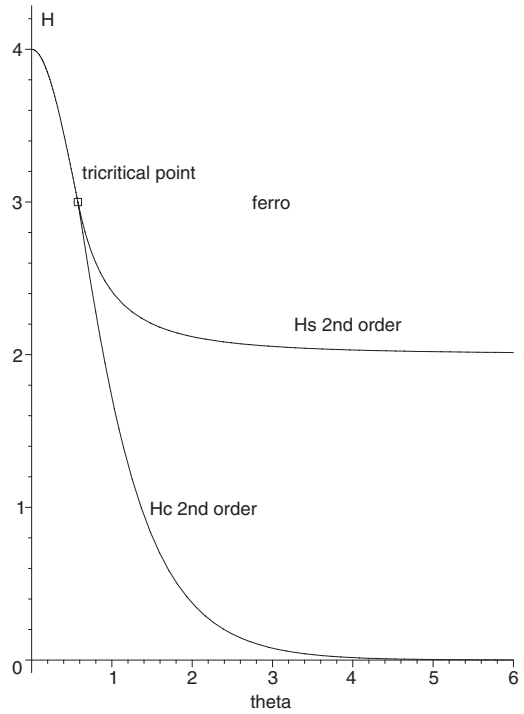

Figure 5. The ground-state phase diagram in the magnetic field–coupling parameter plane $H-\theta$ for the two-chain spin-$\frac{1}{2}$ AF model for $J = 1$. There are three phases: the spin-saturated (ferromagnetic) phase, the commensurate phase (with one Dirac sea for low-lying excitations, see figure 3(a)) and the incommensurate one (with two Dirac seas for low-lying excitations, see figure 3(c)). Two second-order phase transition lines merge at the tricritical point (see figure 3(b)).

be the standard R-matrix of the HC. Here indices α_i and β_i denote states of the spin at site i (physical quantum space), and μ denotes states in the auxiliary space. The 'standard' transfer matrices (row to row from the viewpoint of the statistical two-dimensional problem) have the form of the trace over the auxiliary space of the product of R-matrices. We introduce R-matrices of different type, related to the initial one by an anticlockwise rotation $\bar{R}^{\mu\nu}_{\alpha\beta}(x) = R^{\alpha\beta}_{\nu\mu}(x)$ and $\tilde{R}^{\mu\nu}_{\alpha\beta}(x) = R^{\beta\alpha}_{\mu\nu}(x)$ by clockwise rotation. The transfer matrix $\bar{\tau}(x,\theta)$ can be constructed in a way similar to the case of τ. Then we construct $\check{R}$-matrices for the two-chain model according to the rules of the previous subsections. We substitute $x = -J/2NT$, where N is the Trotter number, and find

$$[\tau(x) \otimes \tau(x+\theta) \otimes \bar{\tau}(x) \otimes \bar{\tau}(x+\theta)]^{N/2} = \mathrm{e}^{-\mathcal{H}_2/T+\mathcal{O}(1/N)}. \tag{37}$$

Hence, the partition function of the two-chain quantum model is identical to the partition function of a two-layer (inhomogeneous) classical vertex two-dimensional model with alternating rows on a square lattice of size $L \times N$

$$Z = \lim_{N\to\infty} \mathrm{Tr}[\tau(x) \otimes \tau(x+\theta) \otimes \bar{\tau}(x) \otimes \bar{\tau}(x+\theta)]^{N/2}. \tag{38}$$

The interactions on the quasi-two-dimensional two-layer lattice are four-spin interactions with coupling parameters depending on $(NT)^{-1}$ and interaction parameters θ. Note that the interactions are homogeneous in each column, but vary from column to column. We study

this system in the limit $N, L \to \infty$ using an approach which is based on a transfer matrix describing transfer in the horizontal direction. The corresponding column-to-column transfer matrices are referred to as QTM (where an external magnetic field is included by means of twisted boundary conditions):

$$\begin{aligned}\tau_{\mathrm{QTM}}(x,\theta) = \sum_{\mu,\nu} \mathrm{e}^{\mu_1 H/T}\mathrm{e}^{\nu_1 H/T} \prod_{i=1}^{N/2} & R^{\mu_{2i-1}\mu_{2i}}_{\alpha_{2i-1}\beta_{2i-1}}(x+\mathrm{i}\theta) \otimes \tilde{R}^{\mu_{2i}\mu_{2i+1}}_{\alpha_{2i}\beta_{2i}}(x-\mathrm{i}\theta) \otimes R^{\mu_{2i-1}\mu_{2i}}_{\alpha_{2i-1}\beta_{2i-1}}(x) \\ & \otimes \tilde{R}^{\mu_{2i}\mu_{2i+1}}_{\alpha_{2i}\beta_{2i}}(x) \otimes R^{\nu_{2i-1}\nu_{2i}}_{\alpha_{2i-1}\beta_{2i-1}}(x) \otimes \tilde{R}^{\nu_{2i}\nu_{2i+1}}_{\alpha_{2i}\beta_{2i}}(x) \otimes R^{\nu_{2i-1}\nu_{2i}}_{\alpha_{2i-1}\beta_{2i-1}}(x-\mathrm{i}\theta) \\ & \otimes \tilde{R}^{\nu_{2i}\nu_{2i+1}}_{\alpha_{2i}\beta_{2i}}(x+\mathrm{i}\theta). \end{aligned} \tag{39}$$

In general all QTMs corresponding to L columns can be different. However, all these operators can be proven to commute pairwise. Therefore, the free energy per lattice site of our model can be calculated from just the largest eigenvalues of QTMs (corresponding to only one eigenstate). The free energy per site F of the quasi-one-dimensional quantum two-chain model is given by only the largest eigenvalue of the QTM Λ_{QTM}

$$F = -\lim_{L\to\infty} \frac{T}{L} \sum_{i=1}^{L} \lim_{N\to\infty} \ln \Lambda_{\mathrm{QTM}}(x)\Lambda_{\mathrm{QTM}}(x+\theta) \tag{40}$$

where $x = -\frac{J}{2TN}$ and the dependence on N is understood implicitly.

By means of a Bethe ansatz we find the eigenvalue of the QTM to be given by $\Lambda_{\mathrm{QTM}} = \Lambda(0)\Lambda(\theta)/2^N$, where $\Lambda(u) = \lambda_1(u) + \lambda_2(u)$ and

$$\begin{aligned} \lambda_1(u) &= \phi_+(u)\phi_-(u-2\mathrm{i})\mathrm{e}^{H/T}\frac{Q(u+2\mathrm{i})}{Q(u)} \\ \lambda_2(u) &= \phi_-(u)\phi_+(u+2\mathrm{i})\mathrm{e}^{-H/T}\frac{Q(u-2\mathrm{i})}{Q(u)}. \end{aligned} \tag{41}$$

Here we have dropped the dependence on x and θ, which are fixed, and consider the dependence on the spectral parameter u explicitly. We have used $\phi_\pm(u) = (u \pm \mathrm{i}x)$ and $Q(u) = \prod_{j=1}^{m}(u - x_j)$. Here $\{x_j\}_{j=1}^{m}$ is the set of rapidities which are subject to the BAE. For the largest eigenvalue one has to take $m = N/2$. However, we shall not solve the BAE directly, but rather shall be interested in the functional properties of the eigenvalue of the transfer matrix. For this purpose we introduce four auxiliary functions $b(u)$, $\bar{b}(u)$, $B(u) = 1 + b(u)$ and $\bar{B}(u) = 1 + \bar{b}(u)$ by

$$\begin{aligned} b(u) &= \lambda_1(u+\mathrm{i})/\lambda_2(u+\mathrm{i}) \\ \bar{b}(u) &= \lambda_2(u-\mathrm{i})/\lambda_1(u-\mathrm{i}). \end{aligned} \tag{42}$$

Then one can straightforwardly check that

$$\begin{aligned} \Lambda(u+\mathrm{i}) = B(u)\lambda_2(u+\mathrm{i}) &= \mathrm{e}^{-H/T}\prod_{\pm}\phi_\pm(u+2\mathrm{i}\pm\mathrm{i})\frac{Q(u-\mathrm{i})}{Q(u+\mathrm{i})} \\ \Lambda(u-i) = \bar{B}(u)\lambda_1(u-\mathrm{i}) &= \mathrm{e}^{H/T}\prod_{\pm}\phi_\pm(u-2\mathrm{i}\pm\mathrm{i})\frac{Q(u+\mathrm{i})}{Q(u-\mathrm{i})}. \end{aligned} \tag{43}$$

Obviously

$$\begin{aligned} b(u) &= \mathrm{e}^{2H/T}\prod_{\pm}\frac{\phi_\pm(u\pm\mathrm{i})}{\phi_\pm(u+2\mathrm{i}\pm\mathrm{i})}\frac{Q(u+3\mathrm{i})}{Q(u-\mathrm{i})} \\ \bar{b}(u) &= \mathrm{e}^{-2H/T}\prod_{\pm}\frac{\phi_\pm(u\pm\mathrm{i})}{\phi_\pm(u-2\mathrm{i}\pm\mathrm{i})}\frac{Q(u-3\mathrm{i})}{Q(u+\mathrm{i})}. \end{aligned} \tag{44}$$

One can see that these auxiliary functions are analytic, non-zero and have constant asymptotic behaviour for the strip $-1 < \mathrm{Im}\, u \leqslant 0$ for $b(u)$ and $B(u)$, and for the strip $0 \leqslant \mathrm{Im}\, u < 1$ for $\bar{b}(u)$ and $\bar{B}(u)$. Introducing $a(u) = b(\frac{2}{\pi}(u+\mathrm{i}\epsilon))$ and $\bar{a}(u) = \bar{b}(\frac{2}{\pi}(u-\mathrm{i}\epsilon))$ (an infinitesimal $\epsilon > 0$), taking the logarithmic derivative of these functions, Fourier transforming the equations, eliminating the functions $Q(u)$ and then inverse-Fourier transforming, we obtain the final set of two nonlinear integral equations. Eventually, we take the limit $N \to \infty$. Proceeding in this way we find for our system the following set of nonlinear integral equations for the 'energy density' functions of spinons a, $\bar{a}$, $A = 1 + a$ and $\bar{A} = 1 + \bar{a}$ in dependence on the spectral parameter u:

$$\int \left[k(u-y)\ln A(y) - k(u-y-\mathrm{i}\pi+\mathrm{i}\epsilon)\ln \bar{A}(y)\right]\mathrm{d}y = \ln a(u) + \frac{v}{T\cosh u} + \frac{v}{T\cosh(u+\theta)} - \frac{H}{2T} \tag{45}$$

with $v = \pi J/2$ and kernel function

$$k(x) = \frac{1}{2\pi}\int \mathrm{d}\omega \frac{\mathrm{e}^{-\mathrm{i}\pi\omega x - \pi|\omega|/2}}{2\cosh(\frac{\pi\omega}{2})}. \tag{46}$$

The corresponding equation for $\bar{a}(u)$ is obtained from equation (45) by exchanging $\mathrm{i} \to -\mathrm{i}$, $H \to -H$ and a, $A \leftrightarrow \bar{a}$, $\bar{A}$. The free energy per site F is given by

$$F(u) = e_0(u) - \frac{T}{2\pi}\int \frac{\ln[A(y)\bar{A}(y)]\,\mathrm{d}y}{\cosh(u-y)} \tag{47}$$

where e_0 is the ground-state energy. The free energy of the total two-chain model is $F = f(\frac{\pi}{2}\theta) + f(0)$. These equations are easily solved numerically for arbitrary magnetic field values and temperatures. Analytic results can be obtained for high and low temperatures, as usual. For high temperatures $T \gg J$ with H/T fixed we find that the free energy of the system is just the energy of $2L$ noninteracting spins $\frac{1}{2}$ in a magnetic field with the magnetization $M^z = L\tanh(H/T)$ [36]. For low temperatures $T \ll J$ the free energy is quadratic in T with the specific heat $C = \pi T/6v$ [36] out of the critical in H region (i.e. for $H \neq H_{\mathrm{c,s}}$). In the critical region the low-T free energy is proportional to $T^{3/2}$ and in the tricritical point it is proportional to $T^{5/4}$.

3. Generalizations of the two-chain model

3.1. Anisotropy

The 'basic' model introduced in the previous section can be (and was) generalized in a straightforward manner. The first generalization is the inclusion of a magnetic anisotropy (here we limit ourselves to the $U(1)$-symmetric uniaxial case, but the generalization to the biaxial case is straightforward, although more sophisticated).

In the uniaxial case we can use the L-operator of an XXZ single chain (the parameter γ characterizes the uniaxial 'easy plane' magnetic anisotropy) to construct the $\check{L}$-operator for two-chain model [34]. The algebra is straightforward [34, 50], hence we should like to emphasize the differences between the isotropic and anisotropic cases. The Hamiltonian of the two-chain uniaxial model can be written as

$$\hat{H}_{2\mathrm{anis}} = \frac{J\sin^2\gamma}{\sinh^2\theta + \sin^2\gamma}\sum_j \frac{\sinh^2\theta}{\sin^2\gamma}\big(\boldsymbol{S}_{1,j}\boldsymbol{S}_{1,j+1} + \boldsymbol{S}_{2,j}\boldsymbol{S}_{2,j+1} - (1-\cos\gamma)(S^z_{1,j}S^z_{1,j+1} + S^z_{2,j}S^z_{2,j+1})\big)$$

$$+2\cosh\theta\left(\boldsymbol{S}_{1,j}(\boldsymbol{S}_{2,j}+\boldsymbol{S}_{2,j+1})+\left(\frac{\cos\gamma}{\cosh\theta}-1\right)(S^z_{1,j}(S^z_{2,j}+S^z_{2,j+1}))\right)$$
$$+\frac{2\cos\gamma\sinh\theta}{\sin\gamma}\left((\boldsymbol{S}_{1,j}-\boldsymbol{S}_{2,j+1})[\boldsymbol{S}_{1,j+1}\times\boldsymbol{S}_{2,j}]\right.$$
$$\left.+\left(\frac{\cosh\theta}{\cos\gamma}-1\right)(S^z_{1,j}-S^z_{2,j+1})[\boldsymbol{S}_{1,j+1}\times\boldsymbol{S}_{2,j}]^z\right). \tag{48}$$

In this case we can use $e_n(x) = \sinh[x + \mathrm{i}(n\gamma/2)]/\sinh[x - \mathrm{i}(n\gamma/2)]$ and $a_n(x) = (\mathrm{i}\sin\gamma/2\pi)(\partial e_n(x)/\partial x)$ in equations (20). The isotropic $SU(2)$-symmetric case can be obtained in the limit $\lambda_\alpha \to \gamma\lambda_\alpha$, $\theta \to \gamma\theta$, $\gamma \to 0$. The 'easy-axis' case is reached by the replacement $\gamma \to \mathrm{i}\gamma$.

The ground-state energy for the nonmagnetic case $H = 0$ is given by [34]

$$E_{2\mathrm{anis}} = LJ\cos\gamma\frac{(2\sin^2\gamma+\sinh^2\theta)}{4(\sinh^2\theta+\sin^2\gamma)} - 2LJ\sin\gamma\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\sinh[(\pi-\gamma)\omega]\cos^2(\omega\theta/2)}{\sinh(\pi\omega)\cosh(\gamma\omega)}. \tag{49}$$

For the 'easy axis' case we have [34]

$$E_{2\mathrm{anis}} = LJ\cosh\gamma\frac{(2\sinh^2\gamma+\sinh^2\theta)}{4(\sinh^2\theta+\sinh^2\gamma)} - 2LJ\sinh\gamma\sum_{m=-\infty}^{\infty}\frac{\cos^2(m\theta/2)}{1+\exp(2m\gamma)}. \tag{50}$$

The dispersion law for a spinon in the case of the critical 'easy-plane' (gapless) situation is [50] (see figure 6 for an illustration)

$$\epsilon(p) = J\frac{\pi}{\gamma}\sin\gamma\sin\left(\frac{p}{2}\right)\sqrt{1-\tanh^2\left(\frac{\pi\theta}{\gamma}\right)\sin^2\left(\frac{p}{2}\right)}. \tag{51}$$

Note that if, for small anisotropy γ, the value of θ is large enough to produce a cusp (minimum) for the dispersion law of a spinon, for larger values of γ a one-maximum dispersion law of the spinon is restored.

In fact the ground-state behaviour of this 'easy-plane' two-chain model in the nonzero magnetic field is similar to the isotropic case [50] with the renormalized values of H_s and $H_\mathrm{c} \approx \pi\sin\gamma/\gamma\cosh(\pi\theta/\gamma)$ and $1 \approx \sinh(\pi\theta_\mathrm{c}/\gamma)$ [50]. This implies three possible phases in the ground state: the spin-saturated (ferromagnetic) phase for $H > H_\mathrm{s}$ with gapped excitations, the commensurate critical phase with one Dirac sea for low-lying gapless excitations for $\theta < \theta_\mathrm{c}$ and $\theta > \theta_\mathrm{c}$, $H < H_\mathrm{c}$, and the incommensurate critical phase with two Dirac seas for low-lying gapless excitations for $\theta > \theta_\mathrm{c}$, $H > H_\mathrm{c}$. The point $H = H_\mathrm{c} = H_\mathrm{s}$, $\theta = \theta_\mathrm{c}$ is the tricritical point.

3.2. Different $L_\pm$ and $J_{1,2}$

What are the changes due to different lengths of the chains $L_1 \neq L_2$ and/or $J_1 \neq J_2$ for $\bar{\mathcal{H}}_2$ and $\tilde{\mathcal{H}}_2$ (and for their anisotropic analogues)? In this case one has the construction of the transfer matrix, for example as $\check{\tau}(u,\theta) = \tau^{J_1}(u)\otimes\tau^{J_2}(u+\theta)$. One can obviously see that the values of the momentum, energy and velocity of a spinon (which was $v = \pi J\sin\gamma/2\gamma$ for the anisotropic case) become functions of $(L_+ - L_-)$ and J_1/J_2 [49, 50]. For example, the Fermi velocity renormalizes as $v \to v[1 + (L_+ - L_-)^2\tanh^2(\pi\theta/2\gamma)/(L_+ + L_-)^2]^{-1}$. This introduces dependences of the critical values θ_c and H_c (as well as of the saturation field H_s) on the difference $(L_+ - L_-)$. Also, the Fermi velocities and Fermi points for finite-size corrections become dependent of this difference. Different coupling constants $J_{1,2}$ for each of chains with the Hamiltonians $\bar{\mathcal{H}}_2$ and $\tilde{\mathcal{H}}_2$ (overall multipliers [49]) produce renormalizations

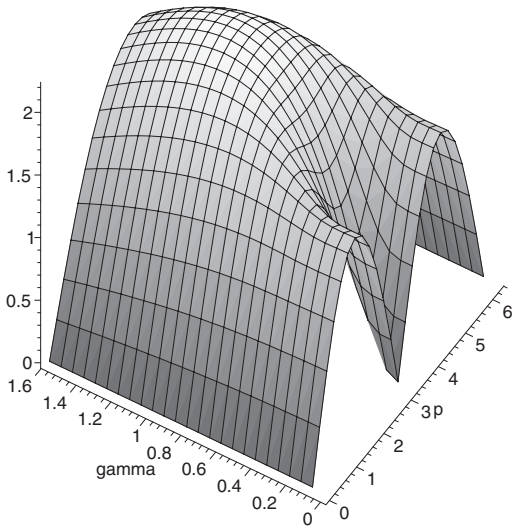

Figure 6. The dispersion law $\varepsilon_{sp}(p)$ for the 'easy-plane' two-chain spin-$\frac{1}{2}$ model as a function of the quasimomentum p and anisotropy γ. The exchange constant J is equal to unity; $\theta = 0.2$, $H = 0$.

similar to the action of $L_+ \neq L_-$. For example, the Fermi velocity renormalizes as $v \to v[1 + (J_1/J_2)^2 \tanh^2(\pi\theta/2\gamma)]^{-1}$ (where $v = \pi J_1 \sin\gamma/2\gamma$).

3.3. Higher-spin generalizations

For the higher-spin generalizations of the Bethe ansatz theory presented in section 2 we can write down the BAE in the form

$$\prod_{\pm} e_{n_\pm}^{L_\pm}(\lambda_\alpha \pm \theta/2) = \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{M} e_2(\lambda_\alpha - \lambda_\beta) \tag{52}$$

where $n_\pm = 2S_\pm$ are the values of spins in each chain. The eigenvalue of the transfer matrix can be written as

$$\Lambda(u) = \prod_{\alpha=1}^{M} \frac{\sinh(u - \lambda_\alpha + \mathrm{i}\gamma\frac{1}{2})}{\sinh(\lambda_\alpha - u + \mathrm{i}\gamma\frac{1}{2})} + \prod_{\pm}\left(\frac{\sinh(u \pm \theta/2)}{\sinh(\mathrm{i}\gamma\frac{n_\pm}{2} - u \mp \theta/2)}\right)^{L_\pm} \times \prod_{\alpha=1}^{M} \frac{\sinh(\lambda_\alpha - u + \mathrm{i}\gamma\frac{3}{2})}{\sinh(u - \lambda_\alpha - \mathrm{i}\gamma\frac{1}{2})}. \tag{53}$$

For the case $S_+ \neq S_-$ the system is the quantum two-chain *ferrimagnet*. For spins $S \neq \frac{1}{2}$ the procedure of the construction of the Hamiltonian is more complicated than for $\frac{1}{2}$ case, because it pertains to the two-chain uniaxial generalization of the Takhtadjan–Babujian model (see [10, 11]). For the simplest case of the isotropic exchange interaction between the spins and between the chains the Hamiltonian has the form [50]

$$\mathcal{H} = \sum_j \big(\theta^2(\mathcal{H}_{S_+,S_+,j_1,j_1+1} + \mathcal{H}_{S_-,S_-,j_2,j_2+1}) + 2(\mathcal{H}_{S_+,S_-,j_1,j_2} + \mathcal{H}_{S_+,S_-,j_1,j_2+1}) + \{(\mathcal{H}_{S_+,S_+,j_1,j_1+1} + \mathcal{H}_{S_-,S_-,j_2,j_2+1}), (\mathcal{H}_{S_+,S_-,j_1,j_2} + \mathcal{H}_{S_+,S_-,j_1,j_2+1})\} + 2\mathrm{i}\theta\big[(\mathcal{H}_{S_+,S_+,j_1,j_1+1} + \mathcal{H}_{S_-,S_-,j_2,j_2+1}), (\mathcal{H}_{S_+,S_-,j_1,j_2} + \mathcal{H}_{S_+,S_-,j_1,j_2+1})\big]\big) \tag{54}$$

where {., .} denotes the anticommutator

$$\mathcal{H}_{S_1,S_2,n,n+1} = \sum_{j=|S_1-S_2|+1}^{S_1+S_2} \sum_{k=|S_1-S_2|+1}^{j} \frac{k}{k^2+\theta^2} \prod_{l=|S_1-S_2|}^{S_1+S_2} \frac{x-x_l}{x_j-x_l} \tag{55}$$

where $x = \vec{S}_{1,n}\vec{S}_{2,n+1}$, and $2x_j = j(j+1)-S_1(S_1+1)-S_2(S_2+1)$ [50]. The Bethe ansatz studies of this model are performed in complete analogy with the above-mentioned case $n_\pm = 1$. The main difference is that for the $SU(2)$-symmetric or uniaxial higher-spin models the ground state corresponds to the filling up of the Dirac seas for spin strings of lengths $n_\pm$ [10, 11]. All previously mentioned characteristic features of the case $n_\pm = 1$ persist. The differences are in the levels of the Kac–Moody algebras in the conformal limit—the conformal anomalies are now equal to $c_\pm = 3n_\pm/(n_\pm+2)$. Now the conformal field theory is a semidirect product of Gaussian ($c = 1$) [71] and $Z(n_\pm)$ parafermion models [72, 73]: the operators identified from the scaling behaviour of states consisting of Dirac sea strings only (obtained from finite-size corrections) are found to be composite operators formed by the product of a Gaussian-type operator and the operator in the parafermionic sector. To find nonzero contributions from parafermions (constant shifts) one can consider the states with strings of other lengths than the Dirac sea present [74–77]. For the scaling dimensions these shifts are $\frac{2r-r^2}{2n+4}$, $r = 1, 2, \ldots$.

For the two-chain ferrimagnet, due to NN and NNN interactions and spin frustration, the ground state is a *singlet (compensated phase)* for $H = 0$, unlike the standard classical ferrimagnets in uncompensated phases (recall that in the Bethe ansatz scheme the ground state is simply determined by filling up all the eigenstates with negative energies). Integral equations that determine the physical vacuum of the systems are similar to equations (21) and (22). They reveal one or several minima of the corresponding distributions of dressed energies and densities with possible negative-energy states, i.e. one or several Dirac seas [50]. An interesting feature due to $S_+ \neq S_-$ is the possible onset of two Dirac seas even for $\theta = 0$. For different values of the spins in chains a phase transition between two different phases is induced by increasing an external magnetic field to some critical value even in the absence of nonzero θ. Here the existence of two Dirac seas can be related to two kinds of different low-lying excitation. They are strings of lengths n_+ and n_-, respectively. In this situation the dispersion laws may be independent (not only factorized as for the case of $S_+ = S_-$). The (new) phase transition at H_c reveals the van Hove singularity of the empty Dirac sea for the longer strings. The spin saturation field H_s is connected with the empty Dirac sea of strings of the smaller length.

3.4. Multi-chain models

One can see from the structure of the Hamiltonians that for the two-chain integrable models the parameter θ characterizes the intra-chain coupling for each chain (or NNN interaction in a single-chain picture). A natural extension of the construction (11) is to introduce the $\check{L}$-operator as the product of $N \times N$ standard L-operators of an HC rather than 2×2 L-operators. Then the whole scheme of the QISM can be applied again; i.e., the problem is exactly solvable, too. In this case it is obvious to introduce the series of $\{\theta_j\}_{j=1}^N$ (for each chain) and to construct the Hamiltonian of the exactly integrable multi-chain (N is the number of chains) spin model. For the simplest case of all, $S = \frac{1}{2}$ isotropic AF chains, the Hamiltonian reads [34–36, 50]

$$\mathcal{H}_N = A\sum_j \left(\left(\prod_{i,k} (\theta_i - \theta_k) \right) \hat{P}_{S_{j,r}S_{j+1,r}} + \sum_{p<q} \frac{\prod_{i,k}(\theta_i-\theta_k)}{(\theta_p-\theta_q)} [\hat{P}_{S_{j,q}S_{j+1,p}},\ \hat{P}_{S_{j,q}S_{j+1,q}} + \hat{P}_{S_{j,p}S_{j+1,p}}] \right.$$

$$+\cdots+\left(\sum_{n=1}^{N}\hat{P}_{S_{j,n}S_{j,n+1}}-\hat{P}_{S_{j,N}S_{j,N+1}}+\hat{P}_{S_{j,N}S_{j+1,1}}\right)\Bigg) \tag{56}$$

where A is the normalization constant (which depends on θ_j) and $\hat{P}_{S_aS_b}\propto \boldsymbol{S}_a\otimes\boldsymbol{S}_b+\text{const}$ is the permutation operator. It is important to point out that for an N-chain system the last term in equation (56) ($\hat{P}_{S_{j,N}S_{j,N+1}}$) is replaced by $\hat{P}_{S_{j,N}S_{j+1,1}}$, i.e. we have *toroidal* boundary conditions in the second space direction rather than periodic ones [36]. Note that in the case of $N\neq 2$ the integrable model corresponds to the pair couplings not only between the NN spins but also to the NNN three-spin, etc, couplings. All these terms are only essential in the quantum mechanics, because in the classical mechanics they are total time derivatives [35] and do not change classical equations of motion. The BAEs have the form

$$\prod_{n=1}^{N}e_1^L(u_m+\theta_n-\theta_1)=\prod_{\substack{k=1\\k\neq m}}^{M}e_2(u_m-u_k) \tag{57}$$

where M is the total number of down spins. The previously considered situation $N=2$ corresponds to the shift of the variables $u_m\to u_m+\theta$ with $\theta_2-\theta_1=-2\theta$. Now $\theta_n-\theta_1$ determines the values of the intra-chain couplings in chain n.

The analysis of the low-temperature thermodynamics of the multi-chain spin system is analogous to the situation of $N=2$ studied above [36, 50]. From the structure of the BAE in the thermodynamic limit one can see that for an N-chain model (for different θ_j) there can exist, generally speaking, N phase transitions of second order in the ground state in an external magnetic field. These are commensurate–incommensurate phase transitions for the quantum multi-chain spin model with different couplings between chains. The values of the critical fields $H_{c_1},\ldots,H_{c_{N-1}}$ and the value of the magnetic field of the transition to the spin-saturated (ferromagnetic) state, H_s, depend on the set of θ_n, i.e. on the intra-chain couplings. The low-lying excitations in the spin-saturated (F) state are gapped, while all other phases have gapless low-lying excitations in the integrable multi-chain spin quantum model. There are also $N-1$ tricritical points at which the lines of the phase transitions H_{c_n} join the line of the spin-saturation (F) phase transition. Naturally, the phase that corresponds to the lowest value of the magnetic field, say $H<H_{c_1}$ for special values of θ_n (the condition is similar to $\theta<\theta_c$ for $N=2$), has a scalar dressed charge in the conformal limit. Hence, in the conformal limit our multi-chain spin model behaves as the level-one WZNW conformal field theory in this phase. In the next phase the multi-chain quantum spin model behaves as the semidirect product of two WZNW models, hence their dressed charges are 2×2 matrices, and so on, until the last gapless phase, which pertains to the semidirect product of N WZNW models with $N\times N$ dressed charge matrices. Note that N in this approach also denotes the number of possible Dirac seas (each of them is connected with the same magnetic field, therefore excitations in each of them are not independent), and, therefore, with one-half of the number of Fermi points. In the limit $N\to\infty$ (i.e. a quasi-two-dimensional spin system) one obtains the (two-dimensional) Fermi surface instead of the set of one-dimensional Fermi points (the latter become distributed more closely to each other with the growth of N) [50]. In this limit the differences between θ_n tend to zero, and this is why the differences between θ_{c_n}, H_{c_n} and also between H_{c_n} and H_s disappear, too. Therefore in this limit only H_s survives. This means that for the quasi-two-dimensional limit of such an integrable model of N coupled quantum spin chains for $N\to\infty$ we expect only *two phases* in the ground state in an external magnetic field: the ferromagnetic gapped one and the gapless phase, which in the conformal limit corresponds to one WZNW model (with a single scalar dressed charge) [50]. The phase transition between these two phases in the ground state in an external magnetic field is of the second order.

3.5. Correlated electron models

Using the applied scheme of the generalization of a standard L-operator for one integrable quantum chain to two and more chains it is possible to construct integrable models not only for spins, but rather for interacting particles with several internal degrees of freedom (e.g. spin). The simplest generalization of the HC is the supersymmetric t–J model [19, 20]. The exactly solvable model of coupled supersymmetric t–J chains was introduced and solved in [78]. Here in the simplest case of two chains the Hamiltonian consists of the terms describing inter- and intra-chain hoppings and exchange couplings, and, in addition, the topological term. The spin part of the topological term is similar to equation (18), but there is also a spin–charge topological term [78]. The latter occurs due to possible charge movement in the supersymmetric t–J model, comparing with the HC. These topological terms measure not only topological spin currents, but also charge currents, i.e. nontrivial topological distributions of the spin *and charge* densities in a multi-chain correlated electron system.

An interesting proposition was made in recent works [79–81]: the authors considered additional 'staggering' of the permutation operators in the construction of $\check{L}$-operators (the spectral parameter of the second chain had an opposite sign to the first chain).

4. Related exactly solvable models

4.1. Lattice models

The well known series of related models are the ones which consider Hamiltonians such as $\bar{\mathcal{H}}_2$ or $\tilde{\mathcal{H}}_2$, i.e. *single chains* in which either shifts of the spectral parameter or different values of site spins were introduced. Here we can refer to [14, 82–86] for regular one-dimensional spin lattices. Naturally, it is possible to introduce an inhomogeneous shift of a spectral parameter and/or different value of spin for a *single* site of a chain (i.e. not regularly). This was realized in [87–91] for a single impurity in quantum spin chains and in [92–94] for random ensembles of disordered impurities in quantum spin chains. In recent years there have been published many papers devoted to exact solutions for one-dimensional correlated electron chains with impurities. For instance, the reader can find the introduction of magnetic [95–99] and nonmagnetic [100–103] single impurities in one-dimensional integrable correlated electron lattice systems (a relatively recent short review of hybridization magnetic impurities in integrable correlated electron systems can be found, e.g., in [104]). Finite concentration of homogeneous impurities in exactly solvable correlated electron models was exactly studied in [105, 106], and random ensembles of disordered impurities in a correlated electron integrable model were introduced in [107].

Another important group of lattice integrable one-dimensional models is similar to the class of integrable models considered in this paper. For these models L-operators (and, hence, monodromies and transfer matrices) were *the same* as for an HC (XXZ chain) or supersymmetric t–J chain; however, the *definition of the Hamiltonian* was different: instead of using the first logarithmic derivative of the transfer matrix as the Hamiltonian, the *first and the second* [108, 109], $\mathcal{H} = (\mathrm{i}/2)J(\partial \ln \tau(u)/\partial u)|_{u=0} + A(\mathrm{i}/2)J(\partial^2 \ln \tau(u)/\partial u^2)|_{u=0}$ (or the *first and the third* [110] $\mathcal{H} = (\mathrm{i}/2)J(\partial \ln \tau(u)/\partial u)|_{u=0} + A(\mathrm{i}/2)J(\partial^3 \ln \tau(u)/\partial u^3)|_{u=0}$) logarithmic derivatives with different coefficients were used (recently the first and the third logarithmic derivatives of the transfer matrix were considered as the Hamiltonian of a correlated electron model [111]). The second logarithmic derivative of the transfer matrix of the HC has a term similar in structure to equation (18), i.e. a three-spin pseudoscalar (though different from our basic model [38], because it pertains to the spin *chirality* rather than to the spin

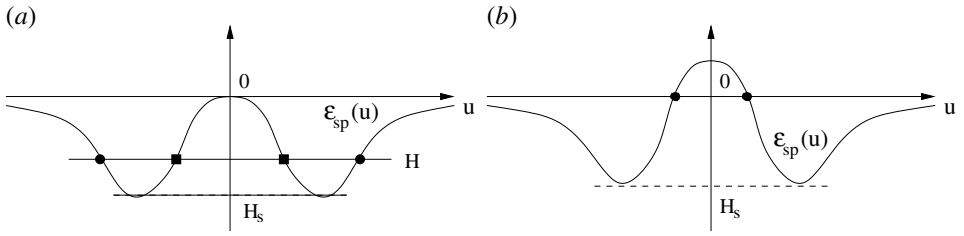

Figure 7. The qualitative picture for the dressed energy of spinons ε_{sp} as a function of the rapidity u for the model with commuting NN and NNN parts of the Hamiltonian with different coupling parameters ($A < 0$ is the coefficient for the NNN part of the Hamiltonian): $A = A_{cr}$ (a), $A < A_{cr}$ (b). In case (b) the coupling A is so strong that for all values of the field H (even $H = 0$) two Fermi seas are created. Filled circles and filled squares denote Fermi points for 'particles' and 'holes' for $H \neq 0$.

antichirality). As for the third logarithmic derivative [110], it consists of a two-spin term describing NNN interactions and four-spin terms. The presence of NN and NNN interactions in this group of models produces features similar to our basic model: the possibility of the two Dirac seas for low-lying excitations, and, as a consequence, magnetic-field-induced phase transitions between commensurate and incommensurate phases. However, there exist important differences between our basic model and these models. For the latter the parts of the Hamiltonian describing NN and NNN interactions *commute*, while for the models considered in this paper the NN and NNN parts of the Hamiltonian *do not commute*. This means, for example, that the average of the three-spin term (the spin chirality) can have its maximal value for some *finite* values of the inter-chain couplings (the latter implies the quantum phase transition governed by a coupling constant A), while for our basic model it is possible only for the *infinite* coupling. The other important difference is that there is a possibility of a *spontaneous* magnetic moment (without any external magnetic field) for the models with commuting NN and NNN parts of the Hamiltonian. This can be seen from the following considerations: Even for zero magnetic field there exists some domain of the coupling parameter A for which there are 'holes' in the distribution of dressed energies of low-lying excitations (see figure 7(b)). Naturally, this implies a spontaneous magnetization (each 'hole' carries a nonzero magnetic moment) [110, 111, 133] and the *first-order* phase transition line $H = 0$ [111]. There is a critical value of the coupling parameter (see figure 8 [111] for the phase diagram) at $H = 0$ (the second-order *quantum phase transition*), which divides commensurate and incommensurate (ferrimagnetic) phases. It turns out that for both classes of models the common feature is the presence of NN and NNN interactions (not chiral terms, which are absent for [110, 111] models) in their Hamiltonians. This produces additional magnetic-field-induced phase transitions between commensurate and incommensurate phases (which seems to be the generic property for these classes of quantum models).

It is worthwhile to mention another important class of exactly integrable multi-chain models (ladders) [112–116]. In fact all these models are just another realization of the well known Uimin–Sutherland model [12, 13]. Here $N - 1$ $SU(N)$ *fields* (different from the usual magnetic field) play the role of inter-chain interactions (along rungs of the ladders). This produces the main difference between this class of models and that considered in this paper: there is a possibility of *gapped* low-lying excitations in the former case even in zero magnetic field.

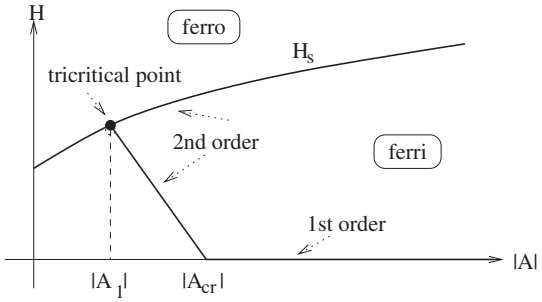

Figure 8. Qualitative depiction of the ground-state phase diagram in the magnetic field–coupling parameter plane H–A for the model with commuting NN and NNN parts of the Hamiltonian. There are three phases: the spin-saturated (ferromagnetic) phase, the commensurate one (for the dressed energies, see, e.g., figure 3(a)) and the incommensurate (ferrimagnetic) one (cf figure 3(c)), with two second-order phase transition curves and the first-order phase transition line (cf figure 7(b)). The latter ends with the second-order phase transition point (figure 7(a)). Two second-order phase transition lines merge at the tricritical point (see figure 3(b)).

4.2. Quantum field theory and impurity models

Two large classes of Bethe ansatz solvable models share similar structure of the BAE with the multi-chain models considered in previous sections. These are (1 + 1) QFTs and Kondo–Anderson impurity models. It is important to emphasize that the determinations of the Hamiltonians (and, hence, energies) are *different* for lattice models and these two classes of integrable models. However, the 'ideology' of the Bethe ansatz consideration remains the same as for the multi-chain models considered in the previous sections. It turns out that for most of the examples of these two classes one needs only single-chain counterparts $\bar{\mathcal{H}}_2$ or $\tilde{\mathcal{H}}_2$.

For the former class, $L_\pm$ corresponds to the numbers of (bare) particles with positive and negative chiralities. For example, for the chiral-invariant Gross–Neveu model [117, 118] we have to put $\gamma \to 0$ (i.e. the $SU(2)$-symmetric case, equivalent to the $SU(2)$-symmetric Thirring model), and $\theta = (1 - g^2)/2g$, where g is the coupling constant of the chiral invariant Gross–Neveu QFT [117]. As for the Lagrange multiplier H, it can play the role of either an external magnetic field, or the chemical potential, or an external topological field, dual to the topological Noether current in QFTs. Here we recall that in the QFT theorists are interested in *physical particles*, which have finite masses (gaps). In the chiral-invariant Gross–Neveu model the gap of the staggered oscillations of the two-chain model plays the role of the physical mass of the particle (spinor) [38, 117]. As for the (gapless) oscillations of the magnetization of the two-chain model, we point out that they are the consequences of the lattice, and play the role of the massless *fermion doublers* of the lattice QFT [119]. The results of the previous section mean that the behaviour of the chiral-invariant Gross–Neveu model (or $SU(2)$-symmetric Thirring model) in an external magnetic field strongly depends on the coupling constant θ (or equivalently on g). For $\theta < \theta_c$ the conformal limit of the QFT corresponds to one level-one WZNW model with the conformal dimension $c = 1$. However, for $\theta > \theta_c$ ($-\theta_c - \sqrt{\theta_c^2 + 4} < 2g < -\theta_c + \sqrt{\theta_c^2 + 4}$) the conformal limit of this QFT in an external magnetic field corresponds to the semidirect product of *two* level-one WZNW models with the conformal dimensions $c = 1$. Two kinds of conformal point for this QFT have been mentioned already [120] in a slightly different context. They were connected with the one WZNW theory or two WZNW theories, coupled via a current–current interaction. This is related to the right–left symmetry of the chiral invariant Gross–Neveu QFT (see, also, [14, 129, 130] for the case of the QFT for the principal chiral field, or nonlinear σ-model). Note that the condition $H > H_c$

in the QFT means that the magnetic field is larger than the mass of the physical particle (colour spinor). In this sense, in the region of the values of a magnetic field $H < H_c$ the results of the QFT (see, e.g., [14, 118]) predict the zero magnetization; however, a different lattice regularization, similar to the lattice scheme used in the previous section, predicts a *nonzero* magnetization of the chiral-invariant Gross–Neveu model in this region. This is the indirect effect of the fermion doublers. In other words, it is connected with the well known mapping of the lattice (e.g., Thirring) model under regularization on two continuum QFTs, either both *bosonic* (a free bosonic QFT and a sine–Gordon one, [121]), or both *fermionic* ones (a free one and a continuum massive Thirring model). There are necessarily two such theories because of the Nielsen–Ninomiya fermion doublers: remember that we have started from a lattice [119].

For other models of QFT the procedure of the *lattice regularization* [122–124] was used. There θ plays the role of the *cut-off* to preserve the mass of the physical particle as finite. For example, for the $U(1)$-symmetric Thirring QFT [119, 125] one can use the results of the previous section with the limit $\theta \to \infty$ taken *after* the thermodynamic limit ($L, L_\pm, M \to \infty$ with their ratios fixed, L being the size of the box). In this case one can obviously obtain the conformal limit of the theory with nonzero physical masses of the particles. Naturally in the limit $\theta \to \infty$ in an external magnetic field only the phase with *two* Dirac seas exists. These Dirac seas correspond to the right- and left-moving particles (with positive and negative chiralities). Actually here our point of view coincides with that of the field theorists. Recently it was shown in [126] that for (1+1) sine–Gordon model the lattice regularization scheme in the 'light-cone' [127] approach gives results similar to ours for the conformal limit of the model. It was shown there that at the ultraviolet fixed point the conformal dimensions of the sine–Gordon model are determined by *two* $U(1)$ charges of excitations (the usual one and the *chiral charge*). The chiral charge pertains to the number of excitations transferred from one Dirac sea to the other, similar to the results of the two-chain model (note that the above-mentioned lattice-regularized sine–Gordon case corresponds in our notations to $\theta \to \infty$, where the integral equations for the particles with the positive and negative chiralities are totally decoupled). We point out here that such a behaviour is not unexpected, because the sine–Gordon QFT belongs to the same class of models as considered in this paper, i.e. its Bethe ansatz description features a shift of rapidities in the BAE in the lattice-regularized theory [126].

Similar new phases with one or two kinds of Dirac sea for similar kinds of low-lying excitation also exist for a number of models in which $n_\pm \neq 1$, for example for $SU(n+1)$ chiral-invariant Gross–Neveu QFT [128]: There $n_+ = n_- = n \neq 1$; for the principal chiral field models (nonlinear σ-models) for CP-symmetric [14, 129] (there $n_+ = n_- \to \infty$) and CP-asymmetric cases [130] (there $n_+ \neq n_-$, $(n_+ + n_-) \to \infty$, $(n_+ - n_-)$ fixed, i.e. the symmetry $SU(2) \times SU(2) \propto O(4)$); and for the $O(3)$-symmetric nonlinear σ-model [131]. The well known fusion scheme was used for the case of a flavour-degenerate situation of the chiral invariant Gross–Neveu QFT, in the absence of flavour fields [132]. Note that, except for the $O(3)$-symmetric case, $\gamma = 0$ everywhere in the above-mentioned models of QFT. This corresponds to *rational* (XXX-like) solutions of the YBE for the two-particle scattering matrices. The total momentum and the energy of the system in the framework of the lattice (local) regularization scheme for some QFT can be written as [119]

$$\begin{aligned} -2a_t E &= \sum_{\pm} \sum_{\alpha=1}^{M} \frac{\partial}{\partial \lambda_\alpha} L_\pm \ln e_{n_\pm}(\lambda_\alpha \pm \theta/2) \\ \mathrm{i}aP &= \sum_{\pm} \sum_{\alpha=1}^{M} L_\pm \ln e_{n_\pm}(\lambda_\alpha \pm \theta/2) \end{aligned} \tag{58}$$

where a and a_t denote the space and time lattice constants, respectively, and their ratio fixes the

velocity of light ('light-cone' approach). The CP-symmetric (chiral invariant) case pertains to the situation in which $n_+ = n_- = n$. The Dirac seas are related to dressed particles with negative energies (strings of length $n_\pm$). The behaviour of the dispersion law for an excited particle in the CP-symmetric case ($n_+ = n_- = n$ and $L_+ = L_-$) is similar to equation (51): for instance, for the chiral-invariant Gross–Neveu QFT and principal chiral field model the rhs of equation (51) must be simply multiplied by $\sin(\pi r/n+1)/\sin(\pi/n+1)$ and the parameter θ in equation (51) has to be replaced by $(n+1)\theta/2$. $r = 1, \ldots, n$ is the rank of a fundamental representation of the $SU(n+1)$ algebra. Note that the limit $(n_+ + n_-) \to \infty$ is the case of the CP-asymmetric case of the QFT for the principal chiral field, i.e. with the *Wess–Zumino term* ($n_+ \neq n_-$) [130]. For the CP-symmetric case one or two Dirac seas of the same type of excitation exist due to nonzero θ, but in the CP-asymmetric case the existence of two Dirac seas can be related to two kinds of different low-lying excitation (particles). They are strings of lengths n_+ and n_-, respectively. In this situation the dispersion laws may be independent (not only factorized as for CP-symmetric cases).

We should like to mention a recent exact study of the $(1+1)$-dimensional WZNW model with a nonzero Wess–Zumino term, whose BAEs share similar features with ours—there is an additional Fermi point (and, hence, a minimum in the dispersion law, which is also connected with NNN interactions [133]). This model has a spontaneous ferrimagnetic moment similar to [108–111].

Another very important class of models, which shares the structure of the BAE with two-chain models (and the structure of Hamiltonians with QFT models), is the class of models of magnetic (Kondo) and Anderson (hybridization) impurities [134–136]. Here the shifts of the spectral parameter (or different value of the spin) are present only at one (*impurity*) site.

5. Imaginary θ

In previous sections we considered two- and multi-chain Bethe ansatz integrable systems with *real* shifts of spectral parameters in transfer matrices. For real θ (and for positive J) the first two terms in the Hamiltonian equation (13) or (14) correspond to the AF NN and NNN interactions, thus to the *spin-frustrated* case. On the other hand, *imaginary* values of θ pertain to the *non-Hermitian Hamiltonian* (three-spin term) and either AF or F NN and NNN interactions. Note that while the Hamiltonian is non-Hermitian for an imaginary θ, the Bethe eigenvalues are *real*. The overall factor $1/(1-|\theta|^2)$ is present in the Hamiltonian for an imaginary θ. This implies the critical points $|\theta| = \pm 1$. At these points the NN and NNN couplings between spins change their signs: for $|\theta| < 1$ the NN coupling is of AF type, while the NNN coupling is of F type, and vice versa for $|\theta| > 1$. It is interesting to note that for a negative NN coupling and positive NNN one the system also formally becomes spin frustrated, while in the opposite case it does not. The presence of these points can be also seen from equations (16), where the first multiplier in the nominator (denominator) of the lhs is totally cancelled by the second multiplier of the denominator (nominator). At these points the effective length of the chain becomes half the size, and the effective site spin becomes $S_{\rm eff} = 1$ in equations (16). The energy and the BAE for these points coincide with those for the Takhtajian–Babujian $S = 1$ single chain [10, 11]. Hence at these points *local spin triplets* are formed from original spins $\frac{1}{2}$. In principle for an arbitrary value of an imaginary θ the model behaves as a mixed chain of effective 'fractional spins' $(1 \pm |\theta|)/2$. We can divide $|\theta| = [|\theta|] + \{|\theta|\}$, where the first term ($[|\theta|]$) denotes the integer part and the second one denotes the fractional part. In this case the ground state pertains to the filling up of two Dirac seas for the strings (bound states) of lengths $1 \pm [|\theta|]$. All other exctations have only positive energies. The Bethe ansatz study in this case is similar to the situation with different spins

for two chains [50] (or a single chain consisting of different spins [85, 86]). The fractional part $\{|\theta|\}$ determines the renormalization of the effective velocity of low-lying excitations $v \to v\cos(\pi\{|\theta|\}/2)$ (cf the renormalization of the Kondo scale, which plays the role of a local Fermi velocity for an integrable chain with a *ferromagnetic impurity* [137]). Now let us consider a nontrivial situation with non-F ground state. The ground-state phase transition to the spin-saturated (F) state is determined by the van Hove singularity of the Dirac sea of the strings of smaller length, while other phase transitions are between the phases with gapless low-lying excitations (commensurate–incomensurate phase transitions).

It turns out that the case of an imaginary θ in the QFT limit corresponds to two coupled WZNW models of the levels $1 \pm [|\theta|]$ with the *nonzero* Wess–Zumino term (the coefficient is equal to $|\theta|$). (It is interesting to note that the case with real θ can be formally considered as the situation with *imaginary* coefficient in front of the Wess–Zumino term.) The conformal anomalies are equal to $c_\pm = 3(1\pm[|\theta|])/(3\pm[|\theta|])$. Now each conformal field theory becomes a semidirect product of a Gaussian ($c = 1$) [71] and $Z(1 \pm [|\theta|])$ parafermion models [72, 73]: The operators identified from the scaling behaviour of states consisting of only Dirac sea strings (obtained from finite-size corrections) are found to be composite operators formed by the product of a Gaussian-type operator and the operator in the parafermionic sector. To find nonzero contributions from parafermions (constant shifts) one can consider the states with strings of other lengths than the Dirac sea present [74, 75]. For the scaling dimensions these shifts are $\frac{2r-r^2}{6\pm2[|\theta|]}$, $r = 1, 2, \ldots$.

6. Beyond the Bethe ansatz integrability

So far we have only considered *integrable* multi-chain quantum models. We have shown that commensurate–incommensurate phase transitions of the second order reveal themselves in an external magnetic field due to intra- and inter-chain interactions (or NN and NNN interactions in a single-quantum-chain picture). The onset of these phase transitions does not depend on the value of the site spins. (They emerge in the presence of the 'easy-plane' magnetic anisotropy, which keeps the system in the critical gapless region.) It is important to note that due to frustrations of interactions between neighbouring spins and the presence of additional terms in Hamiltonians (which violate time-reversal and parity symmetries—spin chiralities or topological currents), for all models considered in the paper one cannot satisfy the conditions of the Lieb–Schultz–Mattis theorem [138] (our models have frustrated but not bipartite lattices). Hence the latter cannot be applied (at least directly) for this class of models. This is why for all considered models there are no spin gaps (except of the trivial one, for the spin-polarized, ferromagnetic, ground state). (We do not consider here the gaps connected with an 'easy-axis' magnetic anisotropy, but rather Haldane-like spin gaps [139] which appear even for the isotropic spin–spin interaction, or gaps of fractional magnetization plateaux [140], see also [141].) It is not clear, however, which features of the behaviour of these integrable models (sometimes referred to as 'fine-tuned' parameters) have to exist for more realistic multi-chain models, and what are the qualitative differences we expect to exist between the integrable multi-chain models and other multi-chain systems. It turns out that one can study the behaviour of the Hamiltonians (13) or (14) with the help of a phenomenological classical approach. Here one replaces quantum spins by classical ones, i.e. the value of each site spin is large, $S \to \infty$. It is easy to observe that the three-spin term (18) does not change classical configurations with minimal energies [51], which are the Néel (*ordered*) state of a single AF chain of length $2L$ for $J2 \ll J1$ (i.e. $\theta \ll 1$; here the NN AF interactions are dominant), the incommensurate spiral phase in the same AF chain with the order parameter (the step of

the spiral) $\cos\phi = -J1/4J2 \equiv -1/2\theta^2$, for $J1 \leqslant 4J2$ ($2\theta^2 \geqslant 1$), and the Néel ordered state of two AF chains of lengths L for $J2 \gg J1$ (i.e. $\theta \gg 1$; here the NNN AF interactions are dominant) [38]. It turns out that such a classical approach does not take into account the quantum nature of the system and produce magnetic ordering—the feature not seen in the exact solution. Hence this phenomenological approach cannot be used for a description of more realistic systems. As we argued above [35], the presence of chiral (topological) terms in the Hamiltonian (which are the total time derivatives and do not change classical equations of motion but rather affect topological properties, like the choice of the θ-vacuum in Haldane's approach) is the reason why the low-lying excitations (and fermiom doublers for lattice QFTs) for this class of models are *gapless* and associated low-energy theories are conformal. Relevant perturbations to this class of integrable models will produce gaps for low-lying excitations. As usual, the algebraic (power-law) decay of correlation functions in the ground state of the models, considered in this paper, determines the *quantum criticality*. This means that, starting from the (conformal) exact solutions, one can argue that the response of more realistic multi-chain systems to perturbations can be evaluated by using perturbative methods, e.g., in a *scaling* framework. For example, let us study the effect of relevant perturbations to the considered Hamiltonians, $\mathcal{H}_r = \mathcal{H} + \delta\mathcal{H}_1$, where one can choose, for example, the standard Heisenberg model for several coupled quantum spin chains as $\hat{H}_r$ and the Hamiltonians of spin chains considered above for the values of θ (in the domain where the three-spin terms are relevant) as $\mathcal{H}$. The correction to the ground-state energy and the excitation gap m (the mass of the particle in the QFT) for the quantum critical system are

$$\begin{aligned} \Delta E &\propto -\delta^{(d+z)/y} \\ m &\propto \delta^{1/y} \end{aligned} \tag{59}$$

respectively, where d is the dimension of the system, and z is the dynamical critical exponent. For the conformally invariant systems studied here one has $d = z = 1$. The application of the standard (hyper-) scaling relations yields $y+x = 2(= z+d)$, where x is the scaling dimension, i.e. $x = 2\Delta_{\mathrm{l}} + 2\Delta_{\mathrm{r}}$, found in section 2 (for the phases with the dressed charge matrices the summation over upper indices is meant). Hence the gap for the low-lying excitations (the mass of the physical particles in the QFT) for perturbed systems will be

$$m \propto \delta^{1/2(1-\Delta_{\mathrm{l}}-\Delta_{\mathrm{r}})}. \tag{60}$$

Note that because of scaling, the behaviour of critical exponents (which are related to the exponents, which we introduced for the integrable multi-chain spin models) in the vicinities of the lines of phase transitions has to be universal. For example, we expect that the spin gap has to exist for the values of the isotropic zigzag inter-chain coupling higher than or of the order of 0.5 ($\theta^2 > 1$) for the two-chain spin-$\frac{1}{2}$ system, where the three-spin couplings are relevant (and the emergence of the spin gap is known exactly [142]).

7. Conclusions

In this article we have reviewed the recent development of the one-dimensional Bethe ansatz solvable multi-chain quantum models. The algebraic version of the Bethe ansatz (the QISM) permitted us to construct new families of integrable Hamiltonians using simple generalizations of the well known constructions of a single-chain one-dimensional quantum model. Our 'basic' model of this class of models (the AF two-chain spin-$\frac{1}{2}$ model with the NN and NNN spin-frustrating interactions) already has revealed the most important properties, which were absent for a single-chain situation. We have shown how the algebra of the QISM works for this model, and what were the most important features of the Hamiltonian (which revealed the topological

properties of the two-dimensional together with the one-dimensional properties). We have considered the solution of the Bethe ansatz for the ground state, low-lying excitations and conformal limit (with the asymptotic behaviour of correlation functions). The phase diagram for the model in the ground state as a function of coupling constant and an external magnetic field has been analysed. We have shown how the spin-frustrating interactions produced the (quantum) phase transitions between commensurate and incommensurate phases (both with gapless excitations), which were absent for single-chain models. We have constructed the thermal Bethe ansatz of this model in the form of the 'QTM'. Then we have considered possible generalizations of our basic model: an inclusion of a magnetic anisotropy, higher-spin representations (including very important case of a one-dimensional quantum ferrimagnet), the multi-chain case (here we have studied the important limiting case of an infinite number of coupled chains, hence a possible scenario of the transition from one to two dimensions in the framework of the Bethe ansatz integrability), internal degrees of freedom of particles at each site (i.e. multi-chain models of strongly correlated electron systems), etc. We have reviewed other groups of multi-chain exactly solvable models (e.g. spin ladders) and explained the similarities and differences between the latter and the considered class of models. We have discussed the common features of the structure of the Bethe ansatz of this class of models and, at first sight, very different integrable models of the quantum field theory and magnetic (Kondo-like) impurities in the electron host. The problem of imaginary values of shifts of a spectral parameter (which pertained to non-Hermitian Hamiltonians) has been solved, too. Finally, we have shown how one can understand the behaviour of non-integrable (less constrained) multi-chain quantum models using exact results obtained for this class of models.

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